Modelling and System Identification – Microexam 1

Prof. Dr. Moritz Diehl, IMTEK, Universität Freiburg, and OPTEC/ESAT-STADIUS, KU Leuven

December 4, 2013, 16:00-17:00, Freiburg

Nachname: Vorname: Matrikelnummer:

Fach: Studiengang: Bachelor ☐ Master ☐ Lehramt ☐ Sonstiges ☐

Please fill in your name above and tick exactly one box for the right answer of each question below.

1. What is the probability density function (PDF) $f_e(x)$ for a normally distributed random variable e with mean μ and variance σ^2? The answer is $f_e(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdots$

(a) ☐ $e^{-\frac{(x-\mu)^2}{2\sigma}}$	(b) ☒ $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	(c) ☐ $e^{\frac{(x-\mu)^2}{2\sigma}}$	(d) ☐ $e^{\frac{(x-\mu)^2}{2\sigma^2}}$

2. What is the PDF of a variable z with uniform distribution on the interval $[a, b]$? For $x \in [a, b]$ it has the value:

(a) ☐ $f_z(x) = (a-b)^2$	(b) ☐ $f_z(x) = (b-a)$	(c) ☒ $f_z(x) = \frac{1}{b-a}$	(d) ☐ $f_z(x) = \frac{x}{\sqrt{b-a}}$

3. What is the PDF of an n-dimensional normally distributed variable z with zero mean and covariance matrix $P \succ 0$? The answer is $f_z(x) = \frac{1}{\sqrt{(2\pi)^n \det(P)}} \cdots$

(a) ☐ $e^{\frac{1}{2}x^T P^{-1} x}$	(b) ☐ $e^{-\frac{1}{2}x^T P x}$	(c) ☐ $e^{\frac{1}{2}x^T P x}$	(d) ☒ $e^{-\frac{1}{2}x^T P^{-1} x}$

4. Regard a random variable $x \in \mathbb{R}^n$ with mean $c \in \mathbb{R}^n$ and covariance matrix $P \in \mathbb{R}^{n\times n}$. For a fixed $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$, regard another random variable y defined by $y = b + Ax$. What is the mean of y ?

(a) ☐ $b^T Ac + c^T Pc$	(b) ☐ $y - b + Ax$	(c) ☒ $b + Ac$	(d) ☐ $Axx^T A^T$

5. Above in Question 4, what is the covariance matrix of y?

(a) ☐ $AP^{-1}A^T$	(b) ☒ APA^T	(c) ☐ $A^T P^{-1} A$	(d) ☐ $c^T Pc$

6. (*) Above in Question 4, what is the mean of the matrix valued random variable $Z = yy^T$?

(a) ☐ $(b + Ac)(b + Ac)^T$	(b) ☐ $bb^T + Acc^T A^T$
(c) ☒ $(b + Ac)(b + Ac)^T + APA^T$	(d) ☐ $bb^T + Acc^T A^T + APA^T$

7. A scalar random variable has the standard deviation a. What is its variance?

(a) ☐ a^{-1}	(b) ☐ $\sqrt{a}$	(c) ☐ a	(d) ☒ a^2

8. A scalar random variable has the variance v. What is its standard deviation?

(a) ☐ v^{-1}	(b) ☒ $\sqrt{v}$	(c) ☐ v	(d) ☐ v^2

9. Regard a random variable $\alpha \in \mathbb{R}$ with zero mean and variance σ^2. What is the mean of the random variable $z = \alpha^2$?

(a) ☐ $\alpha + \sigma$	(b) ☐ $\alpha + \sigma^2$	(c) ☐ σ	(d) ☒ σ^2

10. (*) Regard a random variable $x \in \mathbb{R}^n$ with zero mean and covariance matrix P. What is the mean of $z = x^T x$?

(a) ☒ $\mathrm{trace}(P)$	(b) ☐ $\det(P)$	(c) ☐ $\|P\|_2^2$	(d) ☐ $\|P\|_F^2$

11. What is the minimizer x^* of the convex function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - x$?

(a) ☐ $x^* = -1$	(b) ☒ $x^* = 0$	(c) ☐ $x^* = 1$	(d) ☐ $x^* = \log_e(2)$

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12. What is the minimizer x^* of the convex function $f : \mathbb{R} \to \mathbb{R}$, $\boxed{f(x) = \alpha + \beta x + \gamma x^2}$ with $\gamma > 0$?

(a) ☐ $x^* = \frac{2\beta}{\alpha}$	(b) ☐ $x^* = -\frac{\beta}{\gamma}$	(c) ☒ $x^* = -\frac{\beta}{2\gamma}$	(d) ☐ $x^* = -\frac{\beta}{\alpha}$

13. What is the minimizer of the convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\boxed{f(x) = \|y - \Psi x\|_2^2}$ (with Ψ of rank n) ? The answer is $x^* = \ldots$

(a) ☐ $-(\Psi^T\Psi)^{-1}\Psi^T y$	(b) ☒ $(\Psi^T\Psi)^{-1}\Psi^T y$	(c) ☐ $-(\Psi\Psi^T)^{-1}\Psi^T y$	(d) ☐ $(\Psi\Psi^T)^{-1}\Psi^T y$

14. What is the minimizer of the function $f : \mathbb{R}^n \to \mathbb{R}$, $\boxed{f(x) = \|b + A^T x\|_2^2}$ (with A^T of rank n)? The answer is $x^* = \ldots$

(a) ☐ $-(A^TA)^{-1}A^Tb$	(b) ☐ $(A^TA)^{-1}A^Tb$	(c) ☒ $-(AA^T)^{-1}Ab$	(d) ☐ $(AA^T)^{-1}A^Tb$

15. For a matrix $\Psi \in \mathbb{R}^{N\times d}$ with rank d (and $N \geq d$), what is its pseudo-inverse Ψ^+ ?

(a) ☐ $(\Psi^T\Psi)^{-1}\Psi$	(b) ☐ $(\Psi\Psi^T)^{-1}\Psi$	(c) ☒ $(\Psi^T\Psi)^{-1}\Psi^T$	(d) ☐ $(\Psi\Psi^T)^{-1}\Psi^T$

16. Given a sequence of numbers $y(1), \ldots, y(N)$, what is the minimizer θ^* of the function $\boxed{f(\theta) = \sum_{k=1}^N (y(k) - \theta)^2}$?

(a) ☐ $\frac{N}{\sum_{k=1}^N y(k)}$	(b) ☒ $\frac{\sum_{k=1}^N y(k)}{N}$	(c) ☐ $\frac{1}{N}\sum_{k=1}^N y(k)^2$	(d) ☐ $\frac{1}{N^2}\sum_{k=1}^N y(k)^2$

17. What does "i.i.d." stand for?

(a) ☐ infinite identically disturbed	(b) ☐ infinite identically dependent
(c) ☒ independent identically distributed	(d) ☐ independent identically disturbed

18. Given a sequence of i.i.d. scalar random variables $x(1), \ldots, x(N)$, each with mean μ and variance σ^2, what is the expected value of their arithmetic mean, i.e. of the random variable y_N defined by $\boxed{y_N = \frac{1}{N}\sum_{k=1}^N x(k)}$?

(a) ☒ μ	(b) ☐ $\frac{\mu}{\sqrt{\sigma^2}}$	(c) ☐ $\frac{\mu}{\sigma^2}$	(d) ☐ $\frac{\mu}{N}$

19. In Question 18, what is the variance of the variable y_N ?

(a) ☒ $\frac{\sigma^2}{N}$	(b) ☐ $\frac{\sigma}{N}$	(c) ☐ $\frac{\sigma^2}{N^2}$	(d) ☐ $\frac{\sigma}{N-1}$

20. Given a prediction model $\boxed{y(k) = \theta_1 + \theta_2 e^{x(k)} + \epsilon(k)}$ with unknown parameter vector $\theta = (\theta_1, \theta_2)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $x(1), \ldots, x(N)$ and $y(1), \ldots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta) = \|y_N - \Psi_N\theta\|_2^2$. If $y_N = (y(1), \ldots, y(N))^T$, how do we need to choose the matrix $\Psi_N \in \mathbb{R}^{N\times 2}$?

(a) ☐ $\begin{bmatrix} 1 & x(1) \\ \vdots & \vdots \\ 1 & x(N) \end{bmatrix}$	(b) ☐ $\begin{bmatrix} 1 & -x(1) \\ \vdots & \vdots \\ 1 & -x(N) \end{bmatrix}$	(c) ☐ $\begin{bmatrix} e^{x(1)} & 1 \\ \vdots & \vdots \\ e^{x(1)} & 1 \end{bmatrix}$	(d) ☒ $\begin{bmatrix} 1 & e^{x(1)} \\ \vdots & \vdots \\ 1 & e^{x(N)} \end{bmatrix}$

21. Given an autoregressive affine dynamic system model $\boxed{y(k) = \theta_1 + \theta_2 y(k-1) + \theta_3 y(k-2) + \epsilon(k)}$ with i.i.d. noise $\epsilon(k)$ with zero mean and unknown parameter vector $\theta = (\theta_1, \theta_2, \theta_3)^T$, and given a sequence of N output measurements $y(1), \ldots, y(N)$, we want to compute the estimate $\hat{\theta}_N$ by minimizing the prediction error function $f(\theta) = \sum_{k=3}^N \epsilon(k)^2 = \|y_N - \Psi_N\theta\|_2^2$. If $y_N = (y(3), \ldots, y(N))^T$, how do we need to choose the matrix Ψ_N ?

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$\mathbb{E}(y) = \mathbb{E}(b+Ax) = b + \mathbb{E}(Ax) = b + A\cdot\mathbb{E}(x) = b + Ac$

5. Above in Question 4, what is the covariance matrix of y?

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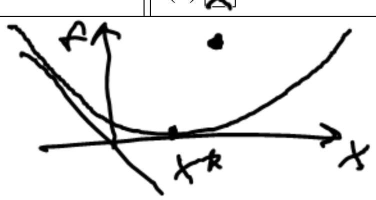

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$\frac{\partial f}{\partial x}(x) = e^x - 1 \overset{!}{=} 0$

$\Longleftrightarrow e^x = 1 \Longleftrightarrow x = \log 1 = 0$

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$\Psi = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

16. Given a sequence of numbers $y(1), \ldots, y(N)$, what is the minimizer θ^* of the function $\boxed{f(\theta) = \sum_{k=1}^N (y(k) - \theta)^2}$?

(a) ☐ $\frac{N}{\sum_{k=1}^N y(k)}$	(b) ☒ $\frac{\sum_{k=1}^N y(k)}{N}$	(c) ☐ $\frac{1}{N}\sum_{k=1}^N y(k)^2$	(d) ☐ $\frac{1}{N^2}\sum_{k=1}^N y(k)^2$

$\Psi^T\Psi = N$

$\boxed{\Psi^+ = \frac{1}{N}\Psi^T}$

17. What does "i.i.d." stand for?

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$x_N = \begin{pmatrix} x(1) \\ \vdots \\ x(N) \end{pmatrix}$

18. Given a sequence of i.i.d. scalar random variables $x(1), \ldots, x(N)$, each with mean μ and variance σ^2, what is the expected value of their arithmetic mean, i.e. of the random variable y_N defined by $\mathbb{E}\left(\boxed{y_N = \frac{1}{N}\sum_{k=1}^N x(k)}\right)$? $= \mu$

(a) ☒ μ	(b) ☐ $\frac{\mu}{\sqrt{\sigma^2}}$	(c) ☐ $\frac{\mu}{\sigma^2}$	(d) ☐ $\frac{\mu}{N}$

Var x_N = $\begin{bmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{bmatrix}$

19. In Question 18, what is the variance of the variable y_N ?

$\exp\left(-\frac{x(k)}{\theta_2}\right)$ NOT LINEAR-IN-PARAMETERS

(a) ☒ $\frac{\sigma^2}{N}$	(b) ☐ $\frac{\sigma}{N}$	(c) ☐ $\frac{\sigma^2}{N^2}$	(d) ☐ $\frac{\sigma}{N-1}$

20. Given a prediction model $\boxed{y(k) = \theta_1 + \theta_2 e^{x(k)} + \epsilon(k)}$ with unknown parameter vector $\theta = (\theta_1, \theta_2)^T$, and assuming i.i.d. noise $\epsilon(k)$ with zero mean, and given a sequence of N scalar input and output measurements $x(1), \ldots, x(N)$ and $y(1), \ldots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta) = \|y_N - \Psi_N\theta\|_2^2$. If $y_N = (y(1), \ldots, y(N))^T$, how do we need to choose the matrix $\Psi_N \in \mathbb{R}^{N \times 2}$?

good!

(a) ☐ $\begin{bmatrix} 1 & x(1) \\ \vdots & \vdots \\ 1 & x(N) \end{bmatrix}$	(b) ☐ $\begin{bmatrix} 1 & -x(1) \\ \vdots & \vdots \\ 1 & -x(N) \end{bmatrix}$	(c) ☐ $\begin{bmatrix} e^{x(1)} & 1 \\ \vdots & \vdots \\ e^{x(1)} & 1 \end{bmatrix}$	(d) ☒ $\begin{bmatrix} 1 & e^{x(1)} \\ \vdots & \vdots \\ 1 & e^{x(N)} \end{bmatrix}$

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good!

(a) ☐ $\begin{bmatrix} 1 & y(1) \\ \vdots & \vdots \\ 1 & y(N-2) \end{bmatrix}$	(b) ☒ $\begin{bmatrix} 1 & y(2) & y(1) \\ \vdots & \vdots & \\ 1 & y(N-1) & y(N-2) \end{bmatrix}$
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S.

$$\operatorname{Var} y$$

$$= \mathbb{E}\,(y - \bar{y})(y - \bar{y})^T$$

$$=$$

$$= A\,\mathbb{E}\,(x - \bar{x})(x - \bar{x})^T A^T$$

$$= A P A^T$$

6.

$$\mathbb{E}(YY^T)$$

$$Y = b + AX$$

$$= \underbrace{(b + Ac)}_{\tilde{b}} + A\underbrace{(X - c)}_{\tilde{X}}$$

$$YY^T$$

$$= (\tilde{b} + A\tilde{X})(\tilde{b} + A\tilde{X})^T$$

$$= \tilde{b}\tilde{b}^T + \underline{\tilde{b}\tilde{X}^T A^T}$$

$$+ \underline{A\tilde{X}\tilde{b}^T} + A\tilde{X}\tilde{X}^T A^T$$

$$\mathbb{E}(YY^T) = \tilde{b}\tilde{b}^T + 0 + 0 + \mathbb{E}(A\tilde{X}\tilde{X}^T A^T)$$

$$= (b + Ac)(b + Ac)^T + A\,\mathbb{E}(\tilde{X}\tilde{X}^T)A^T$$

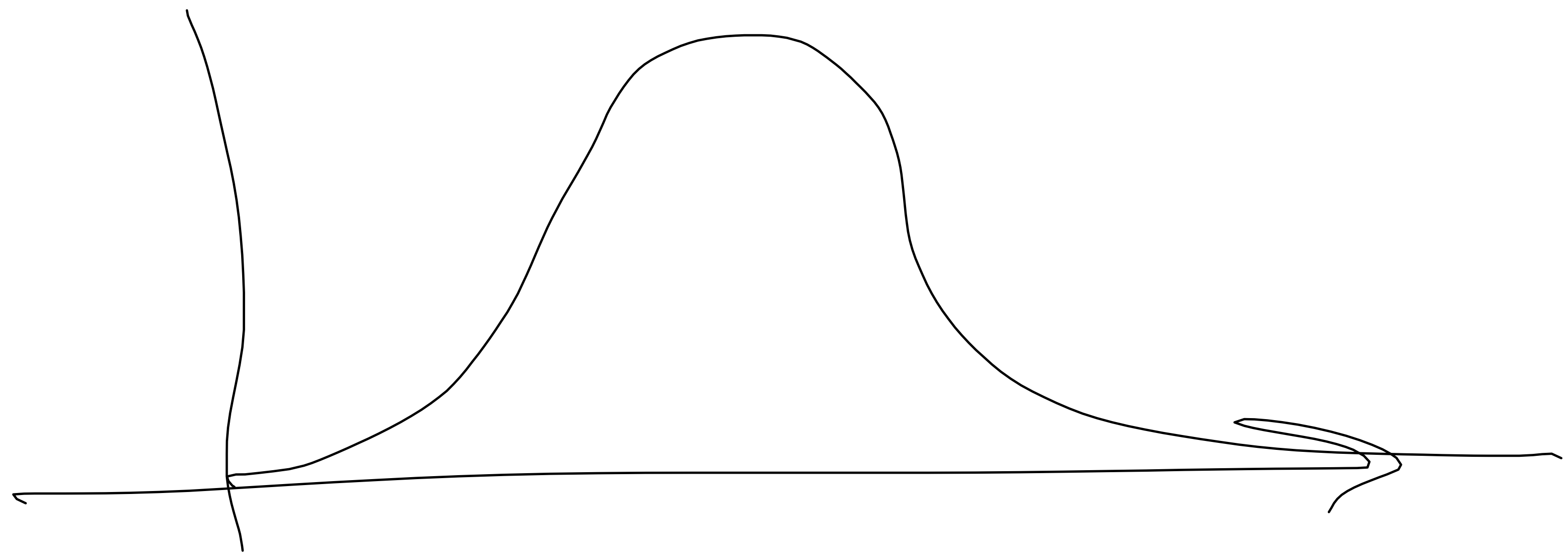
$$= \quad '' \quad + A P A^T$$

9. $\mathbb{E}(a^2)$ mean of a

$$= \mathbb{E}((a - 0)^2)$$

$$= \sigma^2$$

10.



z

$$\mathbb{E}(x^T x) = \mathbb{E}\left(\sum_{i=1}^{n} x_i x_i\right)$$

$$= \sum_{i=1}^{n} \underbrace{\mathbb{E}(x_i x_i)}_{\sigma_i^2} = \sum_{i=1}^{n} \sigma_i^2$$

$$P_{ij} = \left[\mathbb{E}(x_i x_j) \right] \quad \text{(zero mean)}$$

$$= \left[\sigma_1^2 \quad \cdots \quad \sigma_n^2 \right] \qquad \boxed{\mathrm{Tr}(P) = \sum_{i=1}^{n} P_{ii}}$$

zB.

$$X_N = \begin{bmatrix} X(1) \\ \vdots \\ X(N) \end{bmatrix} \qquad \mathbb{V}ar(X_N) = \begin{bmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{bmatrix} = P$$

$$Y_N = \underbrace{\frac{1}{N}[1 \; \cdots \; 1]}_{A} \cdot X_N$$

$$Var(Y_N) = A P A^T$$

$$= \frac{1}{N^2}[1 \; \cdots \; 1] \begin{bmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \sigma^2 = \frac{1}{N} \sigma^2$$