LECTURE 2

[SCHOUKENS, CH 1], CONT

THREE ESTIMATORS:

$$\hat{R}_{LS}(N) = \frac{\frac{1}{N}\sum u(k)\, i(k)}{\frac{1}{N}\sum i(k)\, i(k)}$$

[NOTE: WE DECIDED TO HAVE $u(k)$ AS DEPENDENT VARIABLE & $i(k)$ AS INDEPENDENT VARIABLE (REGRESSOR) (EXPLANATORY VARIABLE)]

$$\hat{R}_{EV}(N) = \frac{\frac{1}{N}\sum u(k)}{\frac{1}{N}\sum i(k)}$$

$$\hat{R}_{SA}(N) = \frac{1}{N}\sum \frac{u(k)}{i(k)}$$

PROBLEM: EVEN FOR $N \to \infty$, DIFFERENT ESTIMATES !

2.1 SIMPLIFIED ANALYSIS [SCHOU. 1.2.2]

ASSUME

$$\underbrace{i(k)}_{\text{MEASUREMENT}} = \underbrace{i_0}_{\text{TRUE VALUE (UNKNOWN)}} + \underbrace{n_i(k)}_{\text{NOISE}} \quad , \quad u(k) = u_0 + n_u(k)$$

ASS 1: NOISE $n_i(k)$ & $n_u(k)$

ARE RANDOM VARIABLES THAT ARE

- MUTUALLY INDEPENDENT,
- identically DISTRIBUTED (i.i.d.)
- ZERO MEAN
- SYMMETRIC DISTRIBUTION
- VARIANCES σ_u^2 AND σ_i^2

DEFINITION: "MEAN" OR "EXPECTED VALUE" OF RANDOM VARIABLE X

$$m = \mathbb{E}\{X\}$$

"VARIANCE"

$$\sigma^2 = \mathbb{E}\{(X-m)^2\}$$

NOTE: "X ZERO MEAN" $\Leftrightarrow$ $\mathbb{E}\{X\} = 0$

"EQUIVALENT TO"

"IF AND ONLY IF"

$$\Downarrow$$

$$\sigma_X^2 = \mathbb{E}\{X^2\}$$

CRUCIAL OBSERVATION: $X(k)$ RANDOM i.i.d. THEN

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X(k) = \mathbb{E}\{X\}$$

ALSO NOTE THAT IF x, y OBEY ASS1, THEN

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x(k)\, y(k) = 0$$

[x INDEP OF y]

ANALYSIS OF $\hat{R}_{LS}(N)$

$$\lim_{N \to \infty} \hat{R}_{LS}(N) = ? = \frac{\lim_{N \to \infty} \frac{1}{N} \sum u(k)\, i(k)}{\lim_{N \to \infty} \frac{1}{N} \sum i(k)\, i(k)}$$

$$= \frac{\lim_{N \to \infty} \frac{1}{N} \sum (u_0 + n_u(k))(i_0 + n_i(k))}{\lim_{N \to \infty} \frac{1}{N} (i_0 + n_i(k))(i_0 + n_i(k))}$$

$$= \frac{\lim \frac{1}{N} \sum [u_0 \cdot i_0 + i_0 \cdot n_u(k) + u_0 \cdot n_i(k) + n_u(k) \cdot n_i(k)]}{\lim \frac{1}{N} \sum [i_0 \cdot i_0 + 2 i_0 \cdot n_i(k) + n_i(k) \cdot n_i(k)]}$$

$$= \frac{u_0 \cdot i_0 + i_0 \underbrace{\lim \frac{1}{N} \sum n_u(k)}_{=0} + u_0 \underbrace{\lim \frac{1}{N} \sum n_i(k)}_{=0} + \underbrace{\lim \frac{1}{N} \sum n_u(k)\, n_i(k)}_{=0}}{i_0 \cdot i_0 + 2 i_0 \underbrace{\lim \frac{1}{N} \sum n_i(k)}_{=0} + \lim \frac{1}{N} \sum n_i(k)^2}$$

THEN WE CAN WRITE

$$\frac{1}{i(k)} = \frac{1}{i_0 + n_i(k)} \approx \frac{1}{i_0}\left(1 - \frac{n_i(k)}{i_0} + \frac{n_i(k)^2}{i_0^2}\right)$$

AND

$$\frac{u(k)}{i(k)} = \frac{u_0 + n_u(k)}{i_0 + n_i(k)} \approx \frac{\left(u_0 + n_u(k)\right)\left(1 - \frac{n_i(k)}{i_0} + \frac{n_i(k)^2}{i_0^2}\right)}{i_0}$$

I.E.

$$\lim_{N\to\infty} \hat{R}_{SA}{}^{N} \approx \lim_{N\to\infty} \frac{1}{N} \sum \frac{u_0 + n_u(k) + u_0 \frac{n_i(k)}{i_0} + u_0 n_i(k)^2}{i_0}$$

$$= \frac{u_0}{i_0} + \underbrace{\lim_N \frac{1}{N} \sum \frac{n_u(k)}{i_0}}_{=0} + \underbrace{\lim \frac{1}{N} \frac{u_0}{i_0^2} n_i(k)}_{=0}$$

$$+ \lim \frac{1}{N} \frac{u_0}{i_0^3} \underbrace{\left(n_i(k)\right)^2}_{G_i^2}$$

$$= \frac{u_0}{i_0} + \frac{u_0}{i_0} \cdot \frac{G_i^2}{i_0^2} = R_0\left(1 + \frac{G_i^2}{i_0^2}\right)$$

OBSERVATION: EVEN IF ~~SEE~~ MEAN OF $\frac{u(k)}{i(k)}$ EXISTS, WE HAVE BIAS

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ANALYSIS OF $\hat{R}_{EV}(N)$

$$\lim_{N\to\infty} \hat{R}_{EV}(N) \overset{?}{=} \frac{\lim \frac{1}{N}\sum u(k)}{\lim \frac{1}{N}\sum i(k)}$$

$$= \frac{u_0 + \lim \frac{1}{N}\sum \underbrace{n_u(k)}_{=0}}{i_0 + \lim \frac{1}{N}\sum \underbrace{n_i(k)}_{=0}} = \frac{u_0}{i_0} = R_0$$

FOR THIS EXAMPLE, $\hat{R}_{EV}$ HAS <u>NO</u> BIAS.

GOOD!

ANALYSIS OF $\hat{R}_{SA}(N)$ "SIMPLE APPROACH"

$$\lim_{N\to\infty} \hat{R}_{SA}(N) = \lim_{N\to\infty} \frac{1}{N}\sum \frac{u(k)}{i(k)}$$

PROBLEM / QUESTION: DOES MEAN OF $\frac{u(k)}{i(k)}$ EXIST?

NOT ALWAYS. DEPENDS ON DISTRIBUTION OF $u(k)$ & $i(k)$.

ASSUME THAT IT EXISTS AND THAT SNR OF $i(k)$ IS HIGH, i.e. $\sigma_i \ll i_0$, OR $n_i(k) \ll i_0$

THEN WE CAN WRITE

$$\frac{1}{i(k)} = \frac{1}{i_0+n_i(k)} \approx \frac{1}{i_0}\left(1-\frac{n_i(k)}{i_0}+\frac{n_i(k)^2}{i_0^2}\right)$$

AND

$$\frac{u(k)}{i(k)} = \frac{u_0+n_u(k)}{i_0+n_i(k)} \approx \frac{\left(u_0+n_u(k)\right)\left(1-\frac{n_i(k)}{i_0}+\frac{n_i(k)^2}{i_0^2}\right)}{i_0}$$

I.E.

$$\lim_{N\to\infty} \hat{R}_{SA}{}^N \approx \lim_{N\to\infty} \frac{1}{N}\sum \frac{u_0+n_u(k) - u_0\frac{n_i(k)}{i_0} + u_0 n_i(k)^2}{i_0}$$

$$= \frac{u_0}{i_0} + \underbrace{\lim_N \frac{1}{N}\sum \frac{n_u(k)}{i_0}}_{=0} + \underbrace{\lim \frac{1}{N}\frac{u_0}{i_0^2} n_i(k)}_{=0}$$

$$+ \lim \frac{1}{N}\frac{u_0}{i_0^3}\underbrace{\left(\frac{n_i(k)}{\cancel{i_0}}\right)^2}_{\to \sigma_i^2}$$

$$= \frac{u_0}{i_0} + \frac{u_0}{i_0}\cdot\frac{\sigma_i^2}{i_0^2} = R_0\left(1+\frac{\sigma_i^2}{i_0^2}\right)$$

OBSERVATION: EVEN IF ~~STR~~ MEAN OF $\frac{u(k)}{i(k)}$ EXISTS, WE HAVE BIAS

2.2 SOME CONCEPTS FROM PROBABILITY THEORY

[LJUNG, APPENDIX I] P. 539/540

$e \in \mathbb{R}$ RANDOM VARIABLE

$f_e(x)$ PROBABILITY DENSITY FUNCTION (PDF)

$P(A)$ = PROBABILITY THAT EVENT A OCCURS

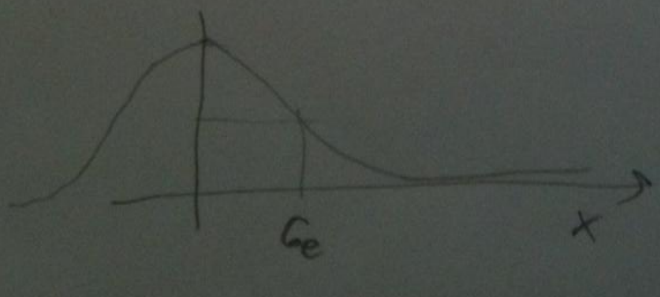
E.G.

$P(a \leq e \leq b)$ = PROB THAT e IS in $[a, b]$

$$= P(e \in [a,b]) = \int_a^b f_e(x)\, dx$$

EXAMPLE : GAUSSIAN (NORMAL) DISTRIBUTION

$$f_e(x) = \frac{1}{\sqrt{2\pi}\,\sigma_e} \exp\left(-\frac{x^2}{2\sigma_e^2}\right)$$



EX [CONT]

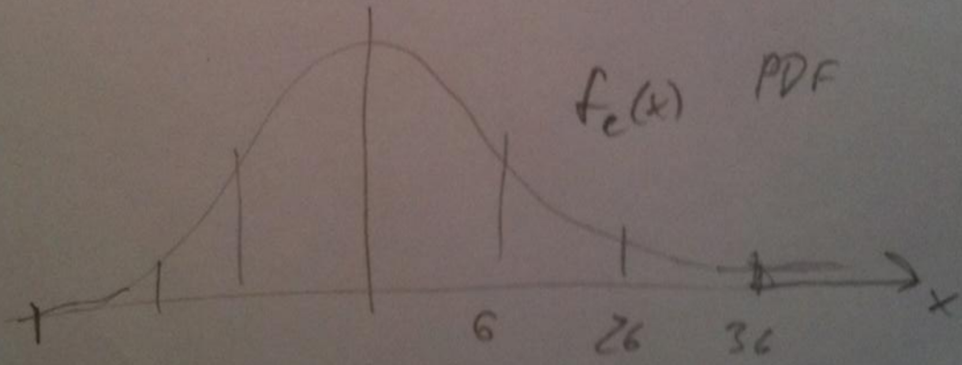
$$P(e \in [-\sigma_e, \sigma_e]) = \int_{-\sigma_e}^{\sigma_e} \frac{1}{\sqrt{2\pi}\sigma_e} e^{-\left(\frac{x^2}{2\sigma_e^2}\right)} dx$$

$$= 0.683 \quad [1\sigma]$$

RECALL "THREE SIGMA RULE"

$$P(e \in [-2\sigma_e, 2\sigma_e]) = 0.955$$

$$P(e \in [-3\sigma_e, 3\sigma_e]) = 0.997$$



DEF: EXP. VALUE
MEAN $\mathbb{E}\{e\} = \int_{-\infty}^{\infty} x \cdot f_e(x) \cdot dx$