LECTURE 3: MULTIDIMENSIONAL RANDOM VARIABLES

$e \in \mathbb{R}^n$ RANDOM $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$

$f_e : \mathbb{R}^n \to \mathbb{R}$, $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto f_e(x_1, x_2, \dots, x_n)$

$f_e(x) \geq 0$

$$\int_{\mathbb{R}^n} f_e(x)\,dx = \int_{-\infty}^{\infty} \int \cdots \int_{-\infty}^{\infty} f_e(x_1, \dots, x_n)\, dx_1 \dots dx_n = 1$$

$$P(x \in B) = \int_B f_e(x)\,dx$$

EXPECTATION (MEAN): $\int_{\mathbb{R}^n} x \cdot f_e(x)\,dx = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbb{E}\{x\} = \begin{bmatrix} \mathbb{E}\{x_1\} \\ \vdots \\ \mathbb{E}\{x_n\} \end{bmatrix} =: m$

COVARIANCE MATRIX:

$$\mathrm{Cov}\ e = \mathbb{E}\{(e-m)(e-m)^T\} = \begin{bmatrix} \mathbb{E}\{(e_1-m_1)^2\} & \mathbb{E}\{(e_1-m_1)(e_2-m_2)\} \\ \vdots & \\ \mathbb{E}\{(e_1-m_1)(e_n-m_n)\} & \dots \end{bmatrix}$$

$$x_1 x_1^T = \left[\begin{array}{c|c|c} x_1x_1 & x_2x_1 & \\ x_1x_2 & x_2^2 & \\ x_1x_3 & \vdots & \dots \\ \vdots & & \\ x_1x_n & x_2x_n & \end{array}\right]$$

NOTE: $P = \mathrm{Cov}\ e$ IS ALWAYS SYMMETRIC & POSITIVE SEMIDEFINITE

3.1

~~DEF~~: MULTIDIMENSIONAL GAUSSIAN / NORMAL DISTRIBUTION

$$e \sim \mathcal{N}(m, P)$$

MEAN $m \in \mathbb{R}^n$

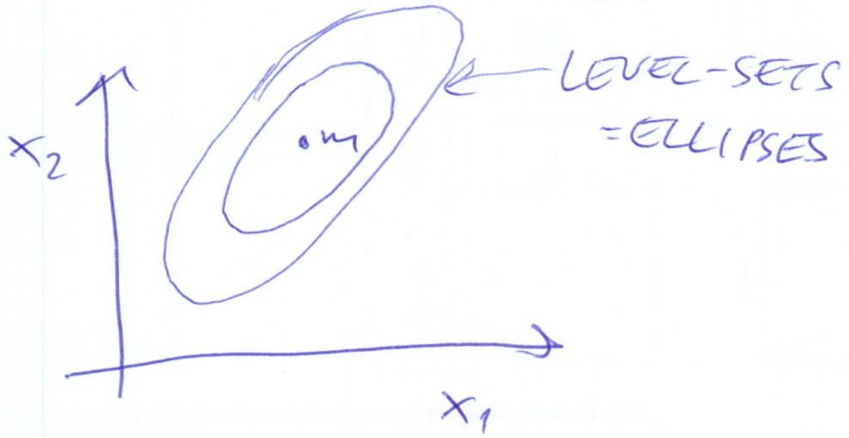
COVARIANCE $\Sigma \in \mathbb{R}^{n \times n}$

$$f_e(x) = \frac{1}{\sqrt{(2\pi)^n \cdot \det P}} \cdot \exp\left(-\frac{(x-m)^T P^{-1} (x-m)}{2}\right)$$

PROPERTIES: $\mathbb{E}\{e\} = m$, $\mathrm{Cov}\{e\} = P$

VISUALIZATION IN 2-D ($n=2$).

VISUALIZATION WITH LEVEL-SETS:



~~PRO~~.

RELATION OF ELLIPSES WITH MATRIX P:

P SYMMETRIC POS. DEF : ORTHONORMAL EIGENVECTORS $v_1, \ldots, v_n$ WITH POSITIVE EIGEN VALUES $\lambda_1, \ldots, \lambda_n$

$$P = V \cdot E \cdot V^T \quad \text{WITH} \quad V = (v_1 | \ldots | v_n), \qquad E = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

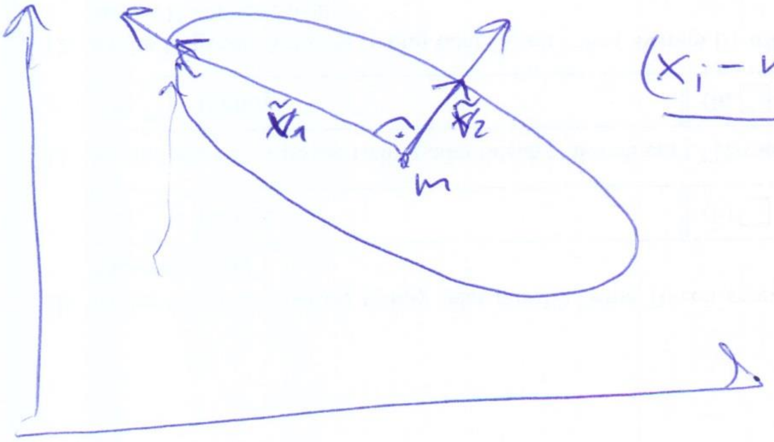
FOR $\tilde{v}_i = \sqrt{\lambda_i} \cdot v_i$ WE HAVE

$$\tilde{v}_i^T P^{-1} \tilde{v}_i = 1 \quad , \text{ i.e. FOR}$$

$x_i = m + \tilde{v}_i$ WE HAVE

$$\frac{(x_i - m)^T P^{-1} (x_i - m)}{2} = \frac{1}{2}$$

i.e.



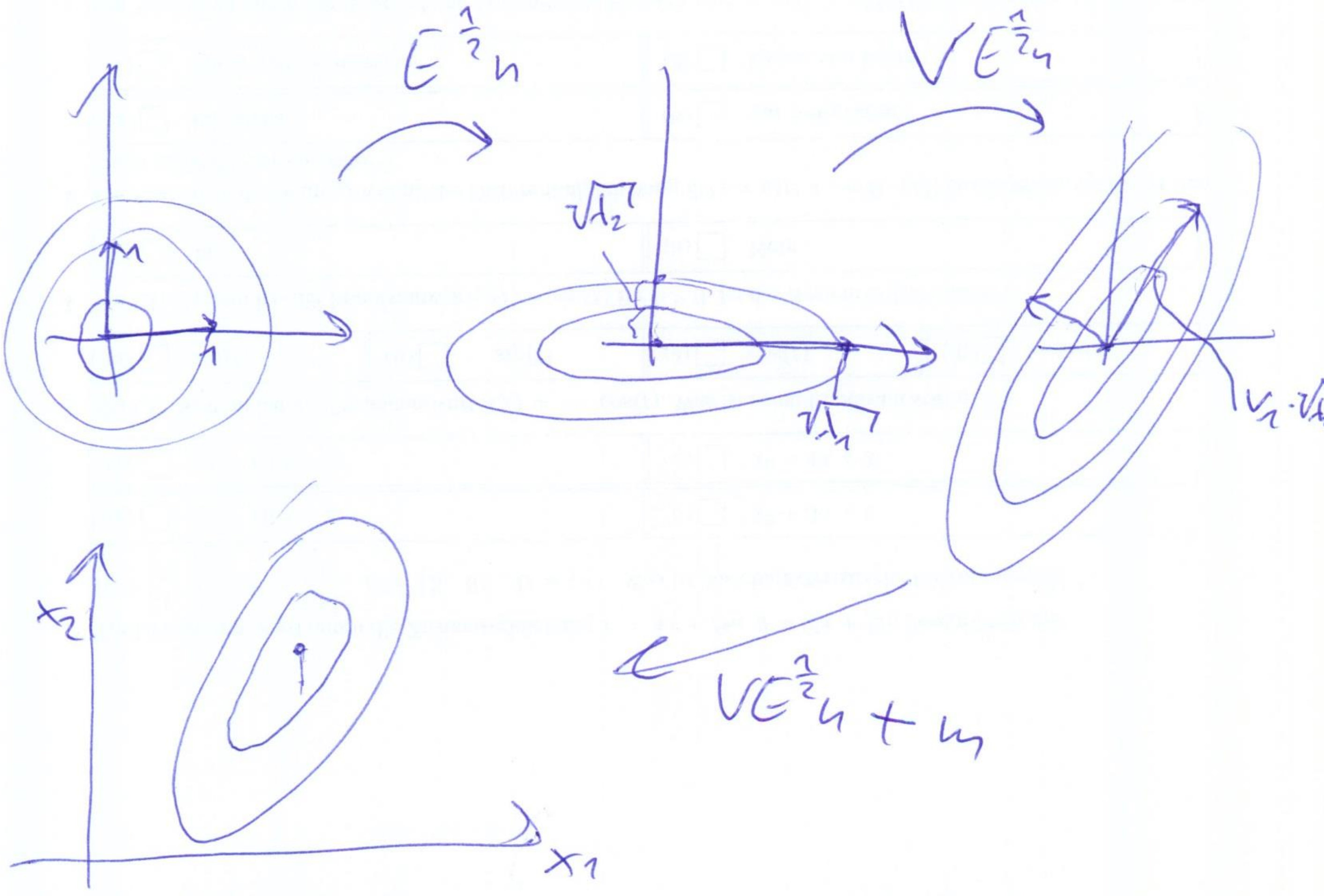
ANOTHER INTERPRETATION:
AFFINE TRANSFORMATION OF
UNIT-NORMAL DISTRIBUTION:

$n \sim \mathcal{N}(0, \mathbb{1})$ WITH PDF $\forall\, n \in \mathbb{R}^n$, RANDOM

$$f_n(x) = \prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}}\ \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2}\right)$$

$$e = \underset{\text{rotate}}{V} \cdot \underset{\text{scale}}{E^{\frac{1}{2}}} \cdot n + \underset{\text{translate}}{m} \qquad E^{\frac{1}{2}} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & \\ & \ddots & \\ & & \lambda_n^{\frac{1}{2}} \end{pmatrix}$$

PDF of n



[LJUNG APP II]

LECT 4: LINEAR LEAST SQUARES

GIVEN: A) ~~A~~ LINEAR MODEL

$$y = \theta^T \varphi \qquad y \in \mathbb{R}, \quad \theta \in \mathbb{R}^d, \quad \varphi \in \mathbb{R}^d$$

B) N MEASUREMENTS $y(k)$ OF y, FOR DIFFERENT REGRESSORS $\varphi(k)$

AIM: FIND "BEST FIT" THAT MINIMIZES LS:

$$f(\theta) = \sum_{k=1}^{N} \left(y(k) - \varphi(k)^T \theta \right)^2$$

$$\frac{\partial f}{\partial \theta_i}(\theta) = \sum_{k=1}^{N} 2 \left(\varphi(k)^T \theta - y(k) \right) \cdot \varphi_i(k)$$

$$\nabla f(\theta) = \begin{bmatrix} \frac{\partial f}{\partial \theta_1} \\ \vdots \\ \frac{\partial f}{\partial \theta_n} \end{bmatrix} = 2 \sum_{k=1}^{N} \varphi(k)\varphi(k)^T \theta - \varphi(k) \cdot y(k)$$

$$= 2 \left(\sum \varphi(k)\varphi(k)^T \right) \theta - 2 \left(\sum \varphi(k) y(k) \right)$$

$$= 2\left(J^T J\right)\theta - 2 J^T Y$$

WITH $Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$, $J = \begin{bmatrix} \varphi(1)^T \\ \vdots \\ \varphi(N)^T \end{bmatrix}$

INDEED:

SOLUTION:

$$\nabla f(\theta) = 0 \iff J^T J \theta = J^T Y$$

$$\iff \theta = \left(J^T J\right)^{-1} J^T Y$$

$$= \underbrace{J^+}_{\text{PSEUDO-INVERSE}} Y$$