Euler-Lagrange Approach to Modelling

Greg and Mario

Outline

Rigid body kinematics

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6. Compute the generalised forces F^q

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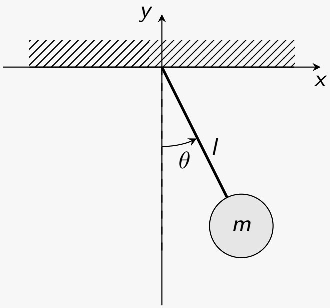

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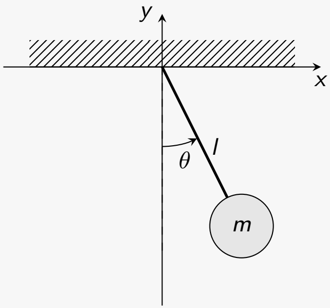

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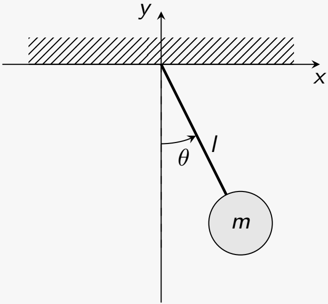

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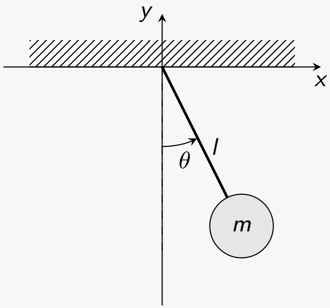

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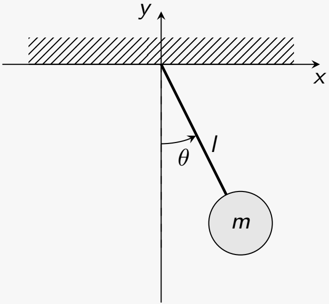

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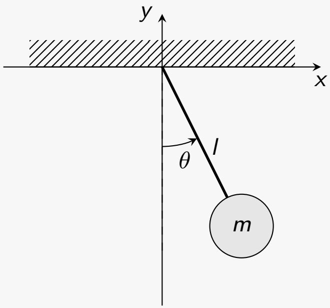

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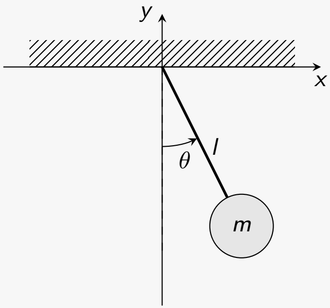

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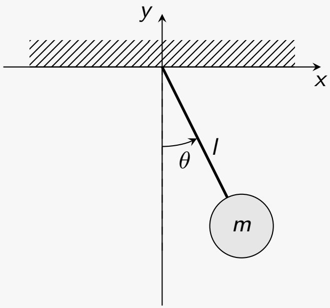

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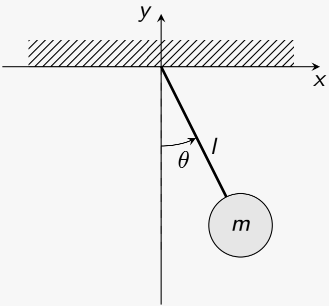

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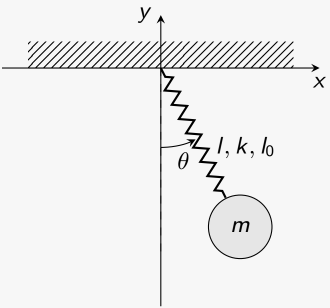

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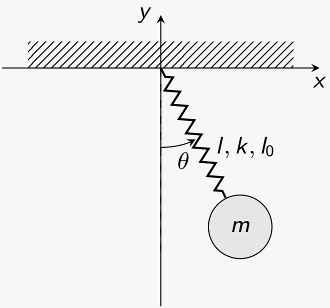

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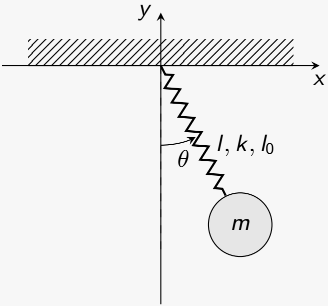

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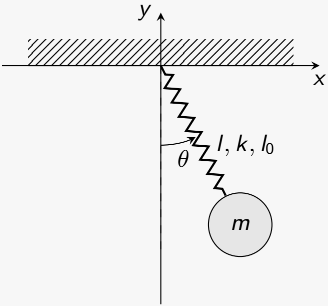

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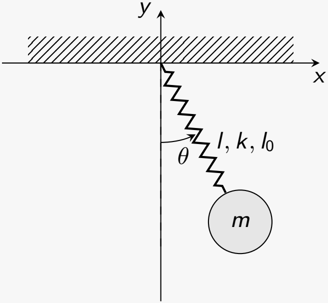

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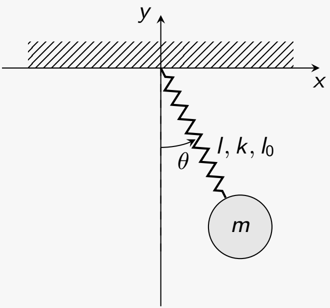

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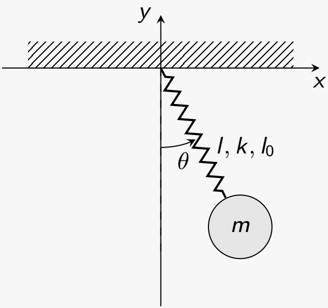

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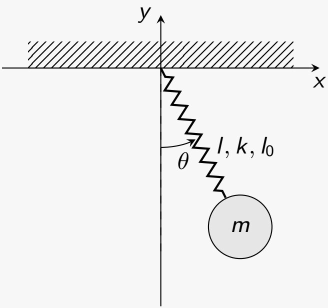

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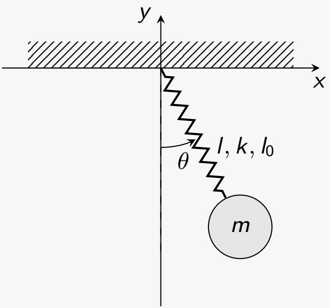

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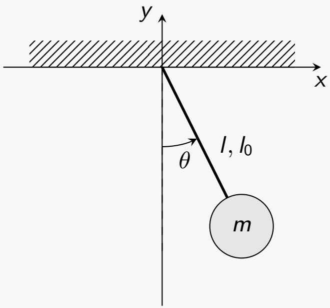

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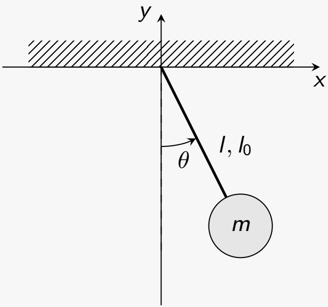

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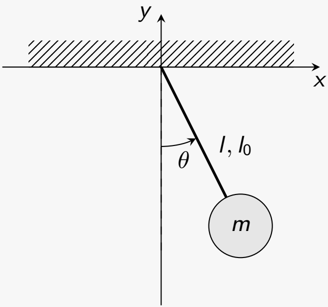

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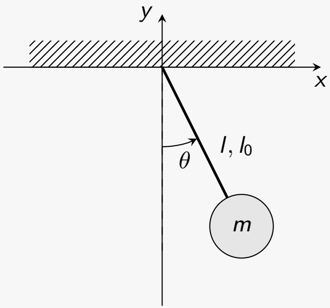

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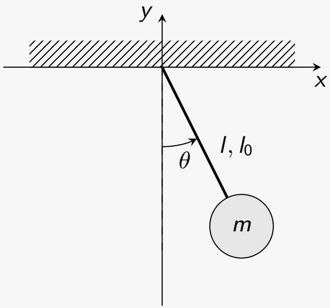

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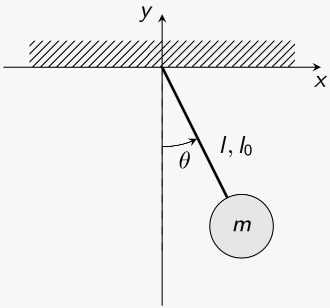

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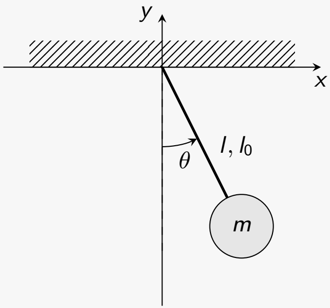

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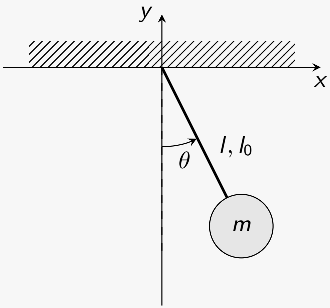

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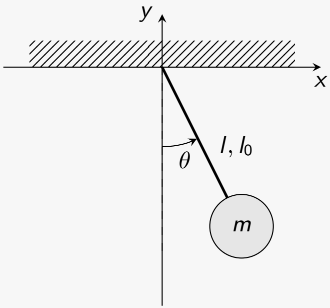
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} m\ddot{l} \\ ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} ml\dot{\theta}^2 + mg\cos\theta - \lambda \\ -mgl\sin\theta \end{bmatrix}$$

Example: Constrained Pendulum



$$q = \begin{bmatrix} l \\ \theta \end{bmatrix}, \qquad p = l\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$\dot{p} = l\dot{\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \dot{l}\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

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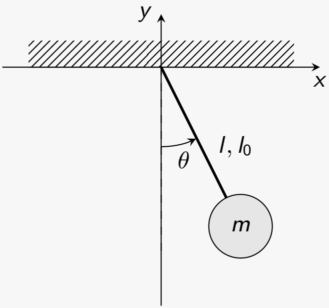
$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \end{bmatrix}$$

Example: Constrained Pendulum



Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

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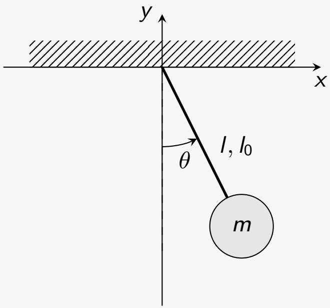

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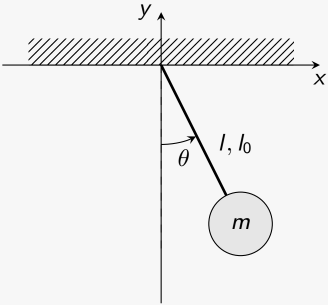
Index reduction:

Example: Constrained Pendulum



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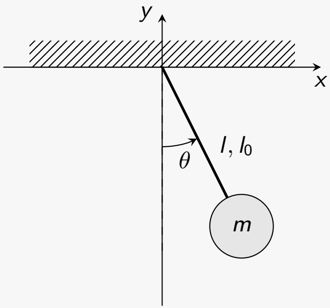
Index reduction:

$\dot{C} = \dot{l},$

Example: Constrained Pendulum



Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Index reduction:

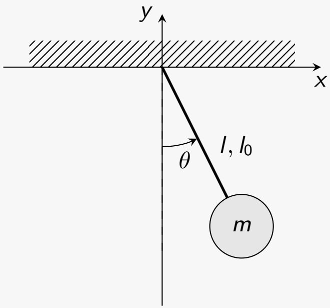
$\dot{C} = \dot{l}, \qquad \ddot{C} = \ddot{l}$

Index-1 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ \ddot{l} \end{bmatrix}$$

$C(t = 0) = 0, \qquad \dot{C}(t = 0) = 0$

Example: Constrained Pendulum



Index-3 DAE

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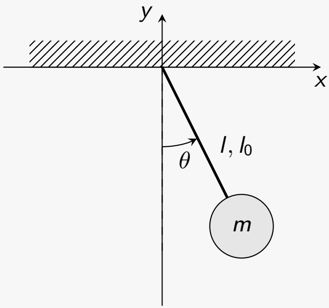
$$C(t=0) = 0, \qquad \dot{C}(t=0) = 0$$

ODE

$$\begin{bmatrix} \ddot{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} \frac{g}{l}\sin\theta \\ ml\dot{\theta}^2 + mg\cos\theta \end{bmatrix}, \quad l = l_0$$

Example: Constrained Pendulum



$\lambda =$ tension in the rod

Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Index reduction:

$\dot{C} = \dot{l}, \qquad \ddot{C} = \ddot{l}$

Index-1 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ \ddot{l} \end{bmatrix}$$

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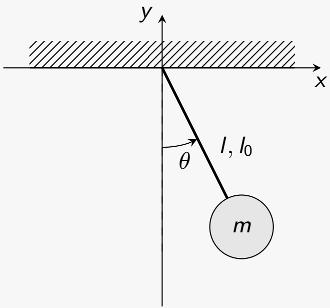
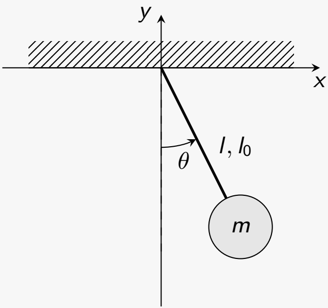
Example: Pendulum in Cartesian Coordinates



$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p = \begin{bmatrix} x \\ y \end{bmatrix},$$

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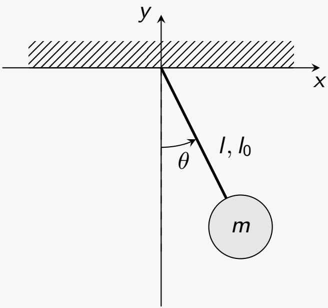

$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

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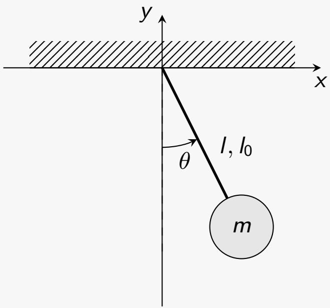

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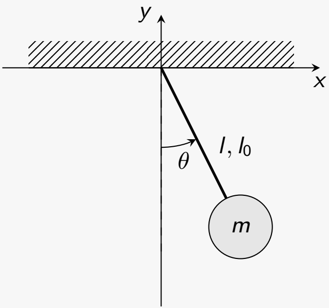

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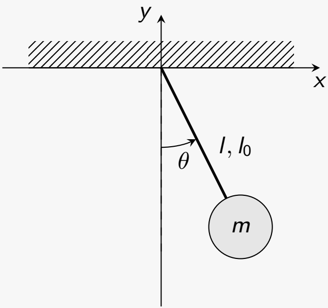

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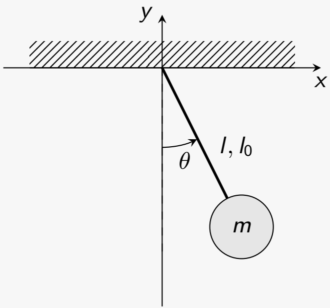

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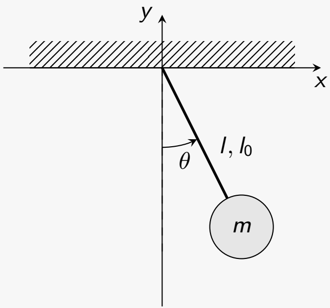

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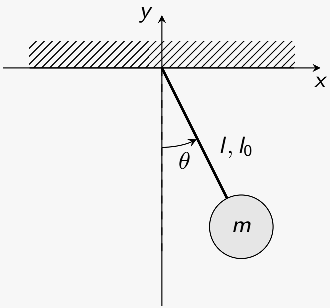

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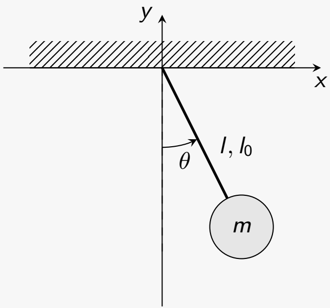
$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



Index-3 DAE

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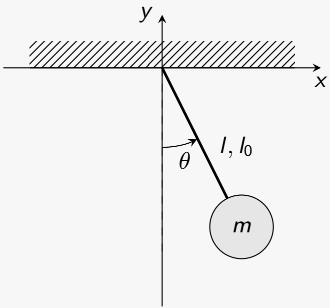

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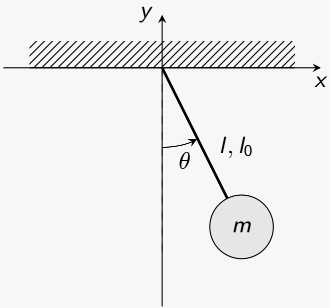
Index reduction:

Example: Pendulum in Cartesian Coordinates



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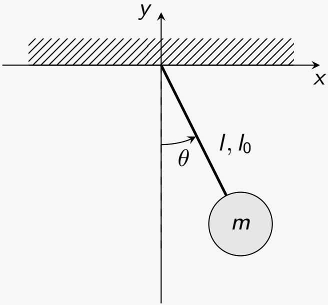
Index reduction:

$$\dot{C} = \dot{x}x + \dot{y}y,$$

Example: Pendulum in Cartesian Coordinates



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Index reduction:

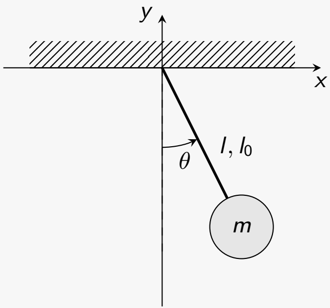
$$\dot{C} = \dot{x}x + \dot{y}y, \qquad \ddot{C} = \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2$$

Index-1 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

$$C(t=0) = 0, \qquad \dot{C}(t=0) = 0$$

Example: Pendulum in Cartesian Coordinates



Index-3 DAE

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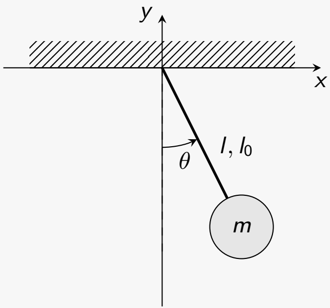
$$C(t=0) = 0, \qquad \dot{C}(t=0) = 0$$

Semi-implicit form

$$\begin{bmatrix} m & 0 & x \\ 0 & m & y \\ x & y & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ mg \\ \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



$$\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \text{rod tension}$$

Index-3 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

Index reduction:

$$\dot{C} = \dot{x}x + \dot{y}y, \qquad \ddot{C} = \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2$$

Index-1 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

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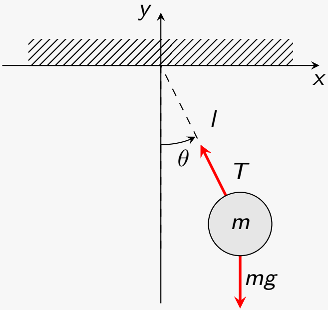
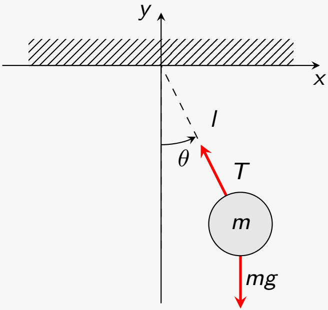
Example: Pendulum in Cartesian Coordinates



Newton's Approach

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} T^x \\ -mg + T^y \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



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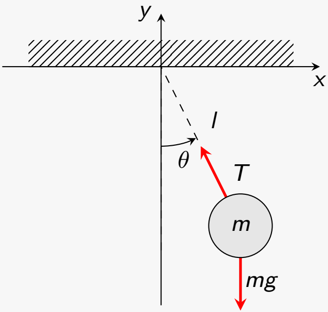
Rod tension:

Example: Pendulum in Cartesian Coordinates



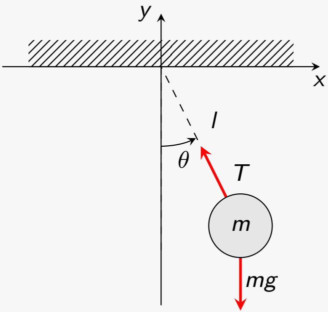
Newton's Approach

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}T^x\\ -mg + T^y\end{bmatrix}$$

Rod tension:

$$T_{\mathrm{g}} = mg\cos\theta = mg\frac{y}{\sqrt{x^2+y^2}} = mg\frac{y}{l}$$

Example: Pendulum in Cartesian Coordinates



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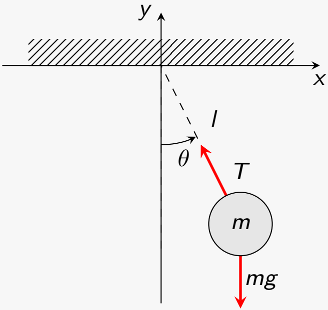
$$T_{\mathrm{c}} = ml\dot{\theta} = m\frac{\dot{x}^2 + \dot{y}^2}{l}$$

Example: Pendulum in Cartesian Coordinates



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Rod tension:

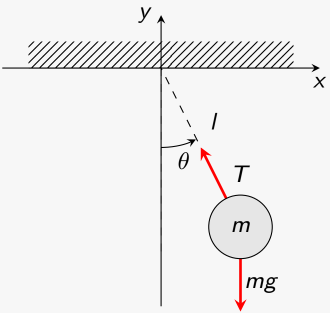
$$T_{\mathrm{g}} = mg\cos\theta = mg\frac{y}{\sqrt{x^2+y^2}} = mg\frac{y}{l}$$

$$T_{\mathrm{c}} = ml\dot{\theta} = m\frac{\dot{x}^2+\dot{y}^2}{l}$$

In the x and y directions:

$$T^x = T\sin\theta = T\frac{x}{\sqrt{x^2+y^2}} = mx\frac{\dot{x}^2+\dot{y}^2-gy}{l^2}$$

Example: Pendulum in Cartesian Coordinates



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$$T_{\mathrm{g}} = mg\cos\theta = mg\frac{y}{\sqrt{x^2+y^2}} = mg\frac{y}{l}$$

$$T_{\mathrm{c}} = ml\dot{\theta} = m\frac{\dot{x}^2+\dot{y}^2}{l}$$

In the x and y directions:

$$T^x = T\sin\theta = T\frac{x}{\sqrt{x^2+y^2}} = mx\frac{\dot{x}^2+\dot{y}^2-gy}{l^2}$$

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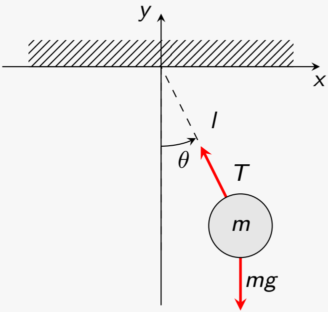
Example: Pendulum in Cartesian Coordinates



$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

Newton's Approach

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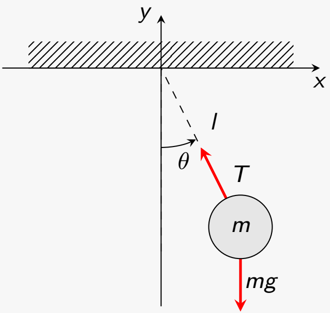

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$$-\lambda x^2 - mgy - \lambda y^2 + m\dot{x}^2 + m\dot{y}^2$$

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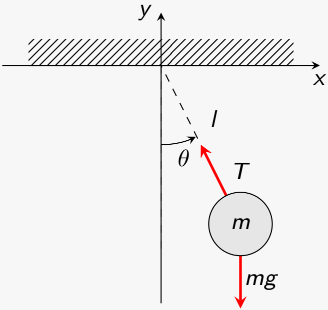

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$$\lambda = m\frac{\dot{x}^2 + \dot{y}^2 - gy}{x^2 + y^2} = m\frac{\dot{x}^2 + \dot{y}^2 - gy}{l^2}$$

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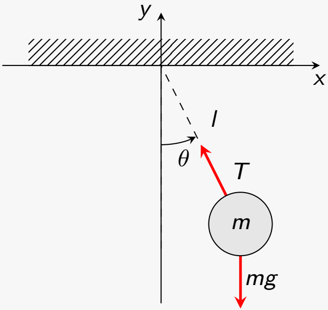

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$$T^x = \lambda x, \qquad T^y = \lambda y$$

Principle of Virtual Works

How to include external forces/torques?

- Generalised coordinates $q \Rightarrow$ generalised forces F^q

Typically, it is natural to express forces F in a coordinate frame $x \neq q$.

How can we compute F^q?

- Virtual displacement: δx
- Principle of Virtual Works: $\delta W := F\delta x = F^q \delta q$
- $x = x(q)$, then: $\delta x = \frac{\partial x}{\partial q}\delta q$

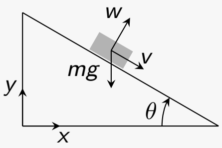
Generalised forces:

$$F^q = F\frac{\partial x}{\partial q}$$

Examples

Examples

Mass on a frictionless inclined plane



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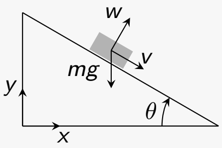

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Examples

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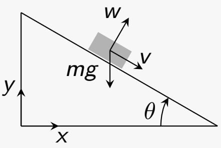

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Examples

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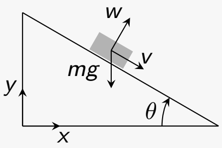

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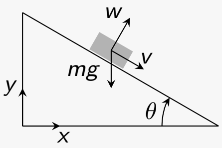
$$q = v, \qquad F\delta y = F^q \delta v,$$

Examples

Mass on a frictionless inclined plane



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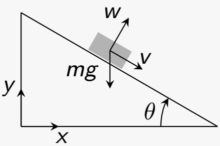

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The generalised force F^q is constant!

Examples

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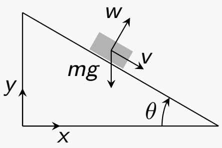

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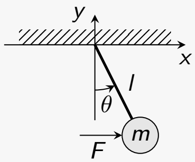
Pendulum with Horizontal Force



Examples

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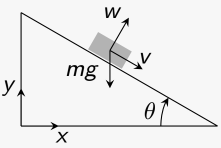

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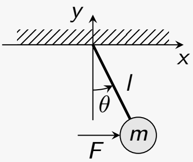

$$q = \theta,$$

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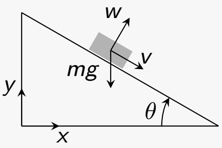

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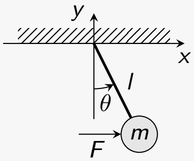

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Examples

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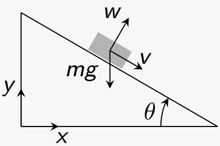

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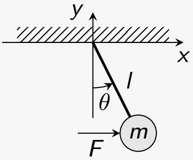

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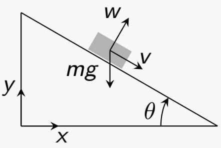

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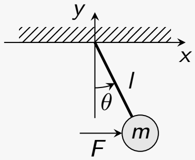

$$q = \theta, \qquad x = l\sin\theta, \qquad F^q = F\frac{\partial x}{\partial \theta} = Fl\cos\theta$$

The generalised force F^q depends on q!

A Nice Interpretation of Euler-Lagrange

A Nice Interpretation of Euler-Lagrange

We will use

$$\frac{\partial v}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x}{\partial q}\right), \qquad \frac{\partial v}{\partial \dot{q}} = \frac{\partial x}{\partial q}$$

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and

$$\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial x}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v\frac{\partial x}{\partial q}\right) - v\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x}{\partial q}\right)$$

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and

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and

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial x}{\partial q} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(v\frac{\partial x}{\partial q}\right) - v\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x}{\partial q}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(v\frac{\partial v}{\partial \dot{q}}\right) - v\frac{\partial v}{\partial q} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial v^2}{\partial \dot{q}}\right) - \frac{1}{2}\frac{\partial v^2}{\partial q}
\end{aligned}$$

Then:

$$m\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial x}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m v^2}{\partial \dot{q}}\right) - \frac{\partial}{\partial q}\left(\frac{1}{2}m v^2\right)$$

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Then:

$$m\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial x}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m v^2}{\partial \dot{q}}\right) - \frac{\partial}{\partial q}\left(\frac{1}{2}m v^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}$$

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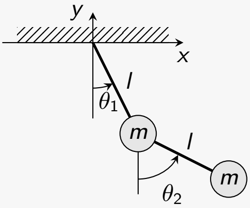
Then:

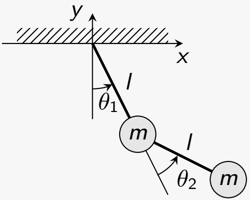
$$m\frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial x}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial mv^2}{\partial \dot{q}}\right) - \frac{\partial}{\partial q}\left(\frac{1}{2}mv^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}$$

The Euler-Lagrange Equations are Newton's Equations projected on the generalised coordinates!

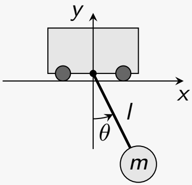
Exercises

- Model a double pendulum in polar and cartesian coordinates (assume $l_1 = l_2 = l$, $m_1 = m_2 = m$)





- Model a pendulum attached to a crane in polar and cartesian coordinates
- Assume to be able to change the length of the crane rod



What we already know

Define

What we already know

Define

- Body frame e

What we already know

Define

- Body frame e
- Rotation $R = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix}$

What we already know

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- Body frame e
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$$L = \underbrace{T}_{T^{\mathrm{tr}}+T^{\mathrm{rot}}} - U - \mathrm{tr}(\Lambda \underbrace{(R^\top R - I)}_{C^{\mathrm{on}}}) - \lambda^\top C(r, R)$$

Kinetic Energy:

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Finally! **The equations of motion**

$$\begin{aligned} m\ddot{r} + \frac{\partial V + \lambda^\top C}{\partial r} &= F^r \\ J\dot{\omega} + \omega \times J\omega + 2\mathcal{P}\left(\frac{\partial}{\partial R}\left(V + \lambda^\top C\right)\right) &= F^{\mathrm{rot}} \\ \dot{R} &= R\Omega \\ C(p, R) &= 0 \end{aligned}$$

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From index reduction (the index-1 DAE imposes $\ddot{C} = 0$):

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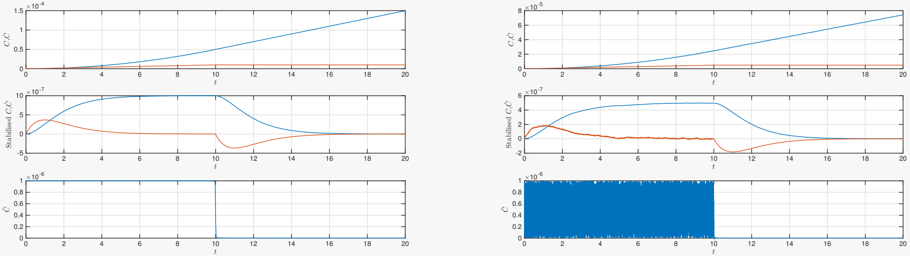

Figure: Invariant simulation with $p = 1$ and $\ddot{C} = 10^{-6}$ or $\ddot{C} = \mathcal{U}(0, 10^{-6})$

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Constraints:

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$$z = \sigma z_{k,j-1} + (1-\sigma) z_{k,j}$$

$$F^{\mathrm{S}}_{k,j-1} = \int_0^1 \sigma F_{k,j}(\sigma)\, \mathrm{d}\sigma$$
$$F^{\mathrm{S}}_{k,j} = \int_0^1 (1-\sigma) F_{k,j}(\sigma)\, \mathrm{d}\sigma$$

Tether Forces

1. Drag force of each tether section:
   - Simple model:

$$F^{\mathrm{S}}_{k,j} = -\frac{1}{2}\rho_{k,j} d_{k,j} l_{k,j} C_{\mathrm{D}} \|v_{k,j}\| v_{k,j}$$
$$v_{k,j} = \frac{\dot{r}_{k,j-1} + \dot{r}_{k,j}}{2} - w\left(\frac{z_{k,j-1} + z_{k,j}}{2}\right)$$

   - Finer model:

$$\mathrm{d}F_{k,j}(\sigma) = F_{k,j}(\sigma)\mathrm{d}\sigma = -\frac{1}{2}\rho_{k,j} d_{k,j} l_{k,j} C_{\mathrm{D}} \|v_{k,j}\| v_{k,j}\,\mathrm{d}\sigma$$
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2. Drag force acting at the generalised coordinates $r_{k,j}$

$$F^{D}_{k,j} = \frac{F^{\mathrm{S}}_{k,j} + F^{\mathrm{S}}_{k,j+1}}{2}$$

Mass Entering/Exiting the Control Volume

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- Is $m = m(t)$?

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F^q + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial m}{\partial \dot{q}} v^2\right) - \frac{1}{2}\frac{\partial m}{\partial q} v^2$$

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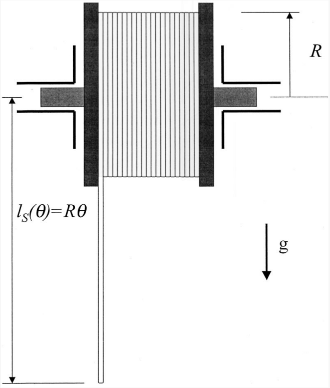
Note that $v = v(t, q, \dot{q})$.

Example: Winch

Control volume: winch + suspended tether



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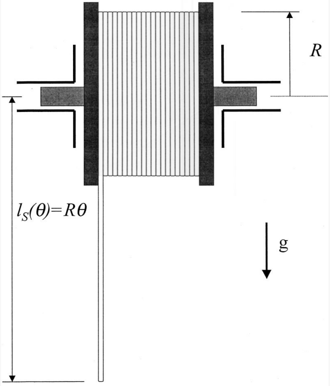

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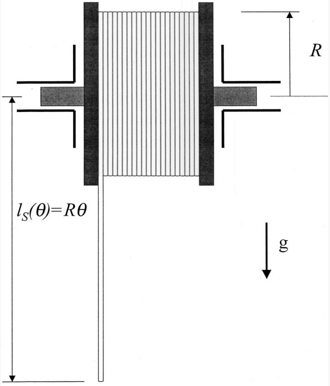

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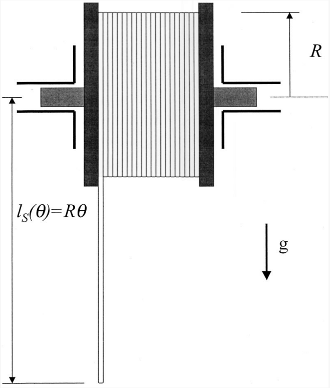

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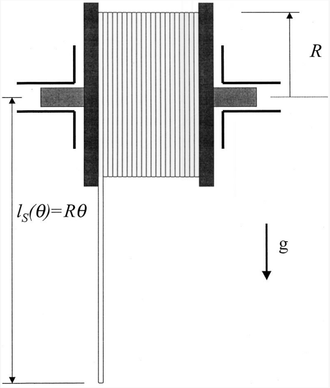

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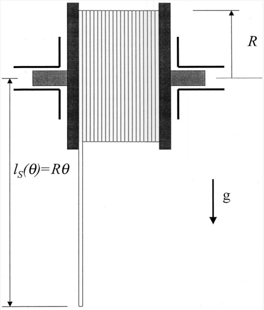

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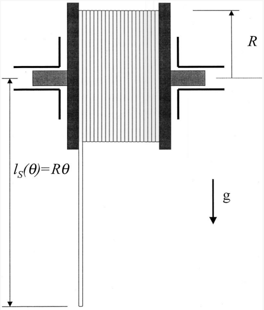

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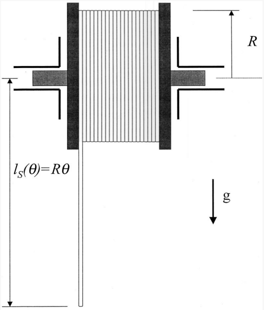

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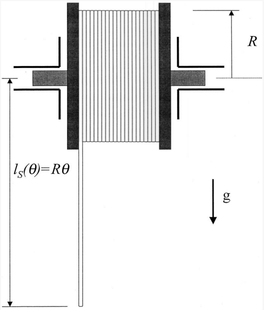

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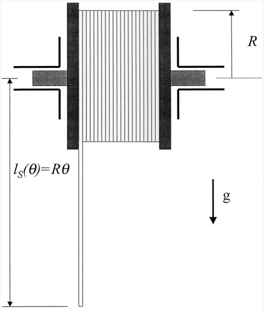

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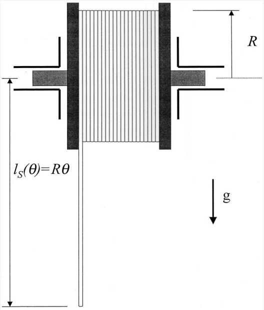

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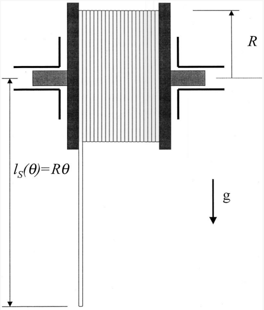

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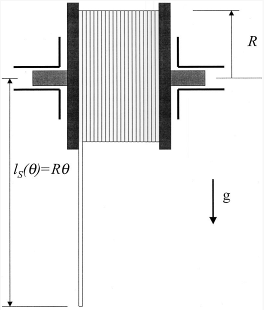

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$$\frac{\mathrm{d}(m_\mathrm{s}v)}{\mathrm{d}t} - F^\mathrm{t} - \dot{m}_\mathrm{s} v_{m_\mathrm{s}} = 0$$

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Tether Models (Pesce2003, Zanon2012, Zanon2013a)

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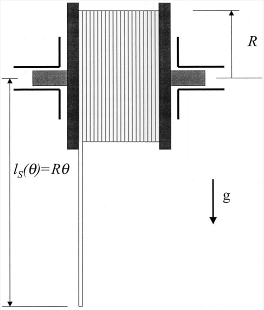

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$$\cancel{\mu R^2\dot{\theta}^2} + \mu R^2\theta\ddot{\theta} - F^{\mathrm{t}} - \cancel{\mu R^2\dot{\theta}^2} = 0$$

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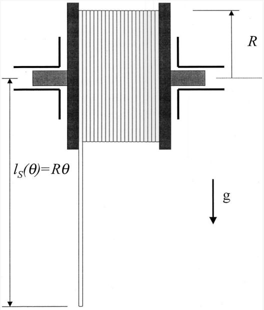

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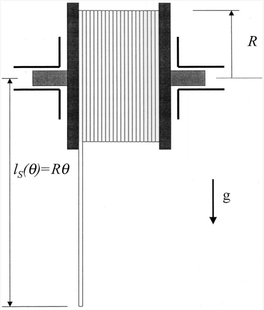

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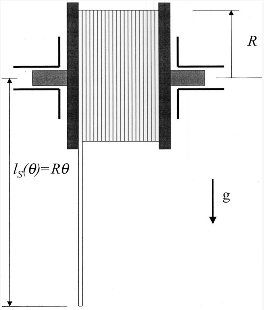

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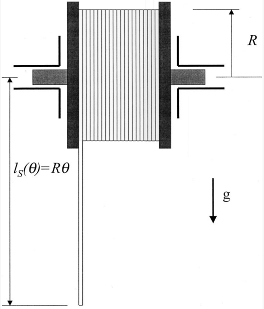
$$(J + mR^2)\ddot{\theta} - \mu g R^2\theta - \frac{1}{2}\mu R^3\dot{\theta}^2 = 0 \qquad \text{Standard EL appr.}$$

Example: Winch

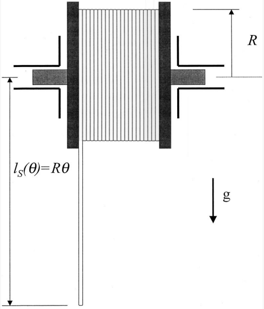
Control volume: tether



$$m = \mu L \qquad m_{\mathrm{s}}(\theta) = \mu R\theta \qquad m_{\mathrm{r}}(\theta) = m - \mu R\theta$$

Example: Winch

Control volume: tether



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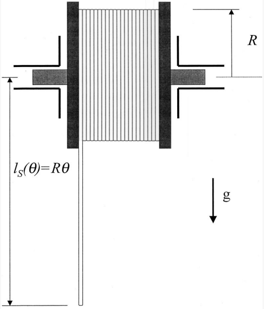
Example: Winch

Control volume: tether



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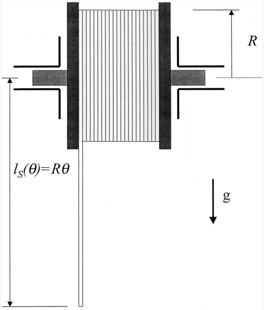
Example: Winch

Control volume: tether



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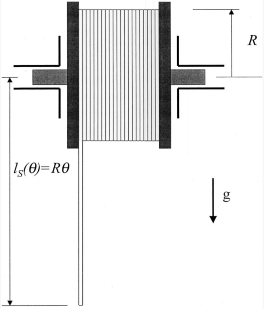
Example: Winch

Control volume: tether



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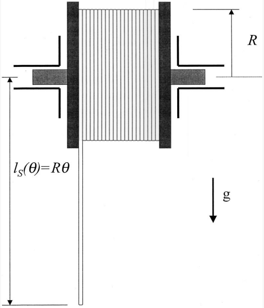
$$\frac{\partial L}{\partial q} = \frac{1}{2}\mu R^3\dot{\theta}^2 - \mu g R^2\theta, \qquad \frac{1}{2}\frac{\partial m_{\mathrm{s}}}{\partial q}v^2 = \frac{1}{2}\mu R^3\dot{\theta}^2$$

Example: Winch

Control volume: tether



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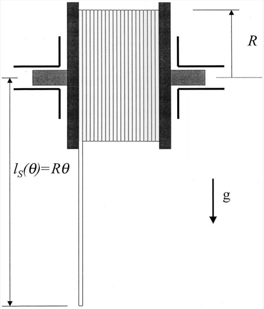
$$\mu R^3\dot{\theta}^2 + \mu R^3\theta\ddot{\theta} - \frac{1}{2}\mu R^3\dot{\theta}^2 = F^q - \frac{1}{2}\mu R^3\dot{\theta}^2$$

Example: Winch

Control volume: tether



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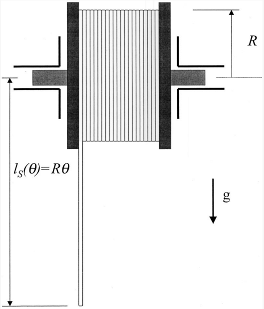
Example: Winch

Control volume: tether



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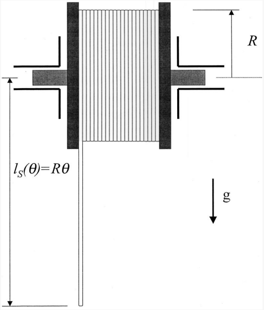
Example: Winch

Control volume: tether



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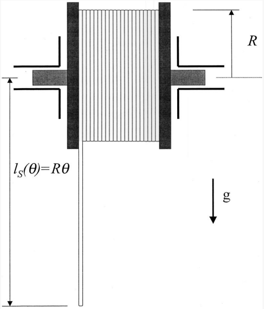
Example: Winch

Control volume: tether



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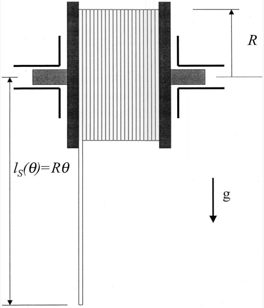
$$RF^\mathrm{t} = -(J + mR^2)\ddot{\theta} + \mu R^3\theta\ddot{\theta}$$

Tether Models (Pesce2003, Zanon2012, Zanon2013a)

Example: Winch

Control volume: tether



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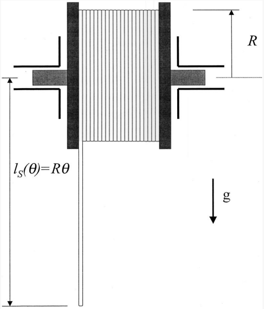
$$-\cancel{\mu R^2\dot{\theta}^2} + (J + mR^2)\ddot{\theta} - \mu R^2\theta\ddot{\theta} + RF^{\rm t} + \cancel{\mu R^2\dot{\theta}^2} = 0$$

$$RF^{\rm t} = -(J + mR^2)\ddot{\theta} + \mu R^3\theta\ddot{\theta}$$

$$(J + mR^2)\ddot{\theta} - \mu g R^2\theta = 0$$

Example: Winch

Control volume: tether



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$$(J + mR^2)\ddot{\theta} - \mu g R^2\theta + \frac{1}{2}\mu R^3\dot{\theta}^2 = 0$$

Standard EL appr.

Exercises

1. Model a single kite without rotational dynamics (pointmass model) [Gros2012a, Zanon2013]
2. Model a single kite with rotational dynamics (but no bridle) [Gros2012f, Zanon2013c]
3. Model a single kite with bridle [Gros2013b]
4. Model a dual kite system [Zanon2014a]