Euler-Lagrange Approach to Modelling

Greg and Mario

Outline

1. Euler-Lagrange Equations
2. Modelling the Rotation Dynamics (Gros2012f,Gros2013b)
3. Baumgarte Stabilisation (Gros2012f)
4. Tether Models (Pesce2003, Zanon2012, Zanon2013a)
5. Kite Models (Gros2013b)

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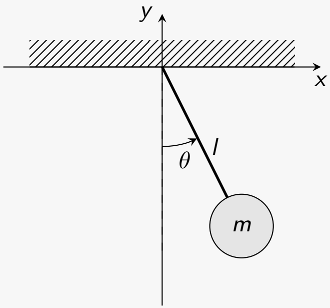

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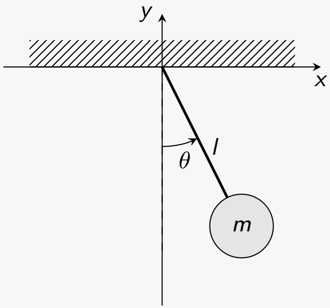

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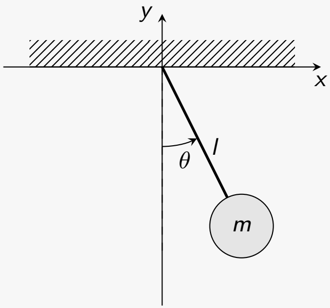

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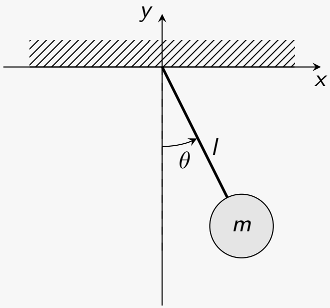

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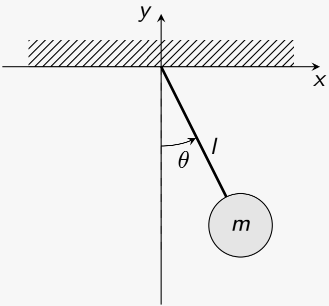

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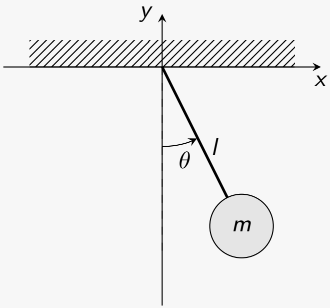

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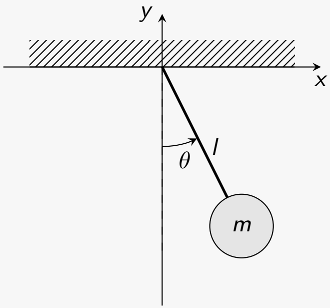

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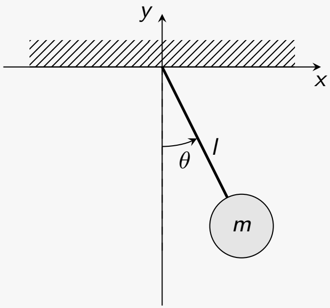

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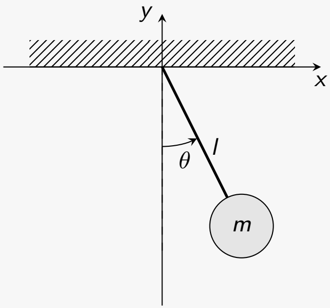

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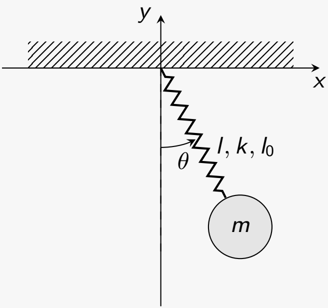

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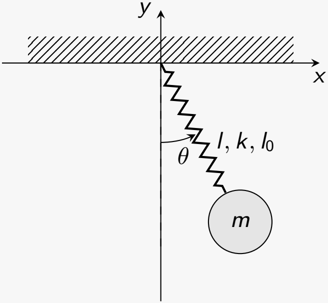

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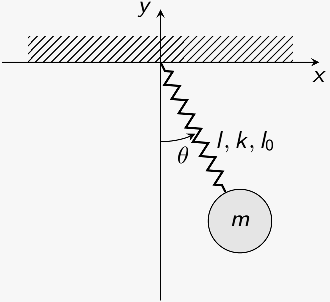

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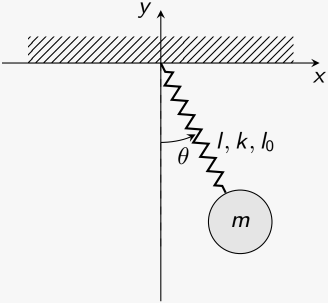

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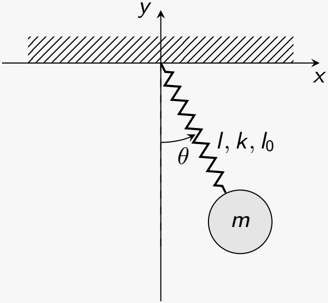

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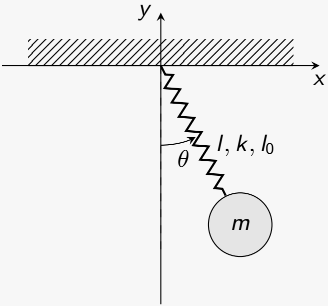

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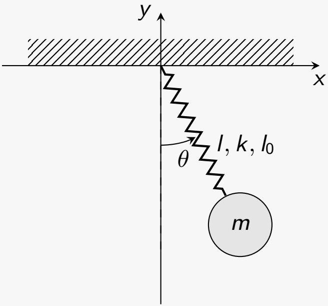

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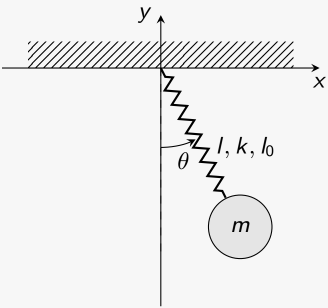

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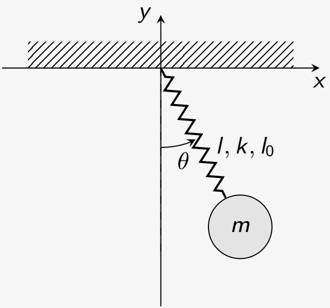

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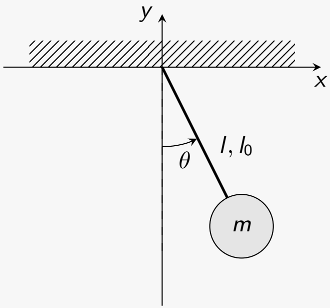

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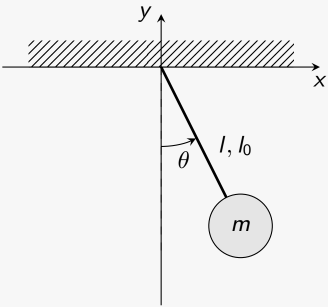

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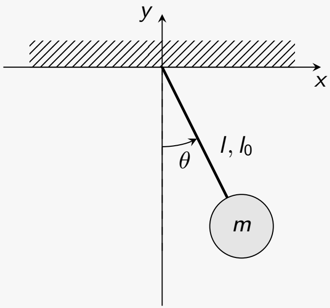

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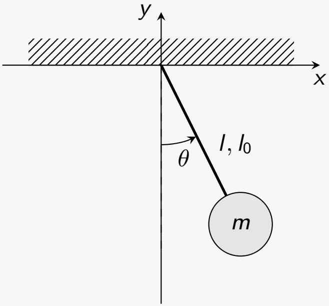

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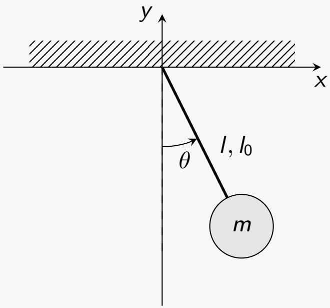

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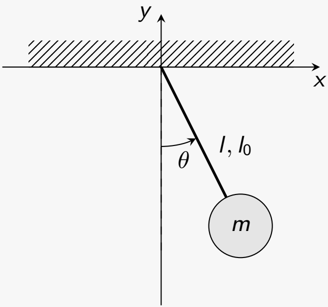

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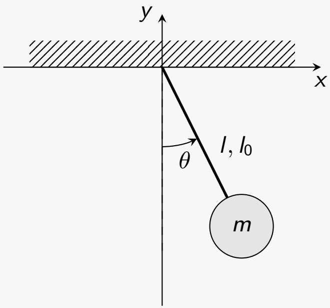

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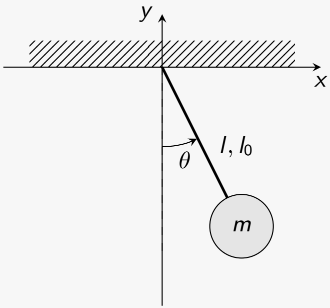

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$$\dot{p} = l\dot{\theta} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \dot{l} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

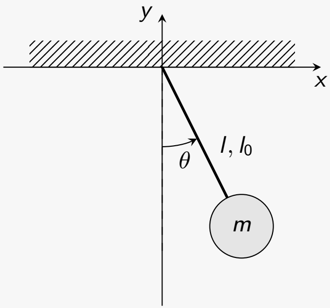
$$T = \frac{1}{2} m \dot{p}^\top \dot{p} = \frac{1}{2} m(l^2\dot{\theta}^2 + \dot{l}^2), \quad U = -mgl\cos\theta$$

$$C = l - l_0, \qquad L = T - U - \lambda C$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} m\ddot{l} \\ ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} ml\dot{\theta}^2 + mg\cos\theta - \lambda \\ -mgl\sin\theta \end{bmatrix}$$

Example: Constrained Pendulum



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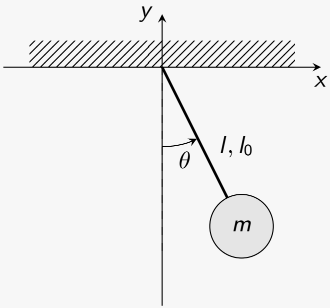
$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \end{bmatrix}$$

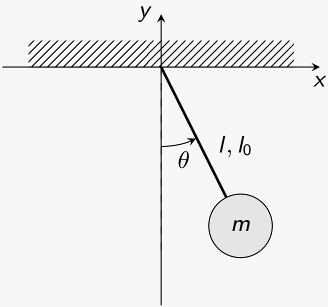
Example: Constrained Pendulum



Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Example: Constrained Pendulum

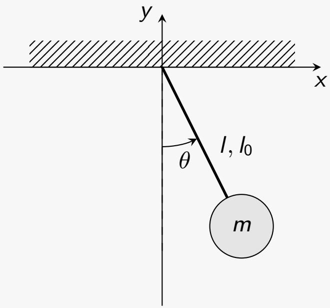


Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Index reduction:

Example: Constrained Pendulum



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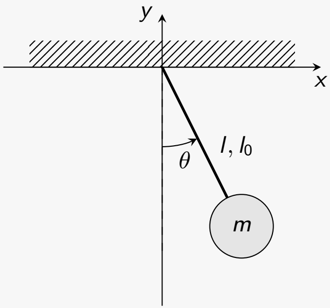
Index reduction:

$\dot{C} = \dot{l},$

Example: Constrained Pendulum



Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Index reduction:

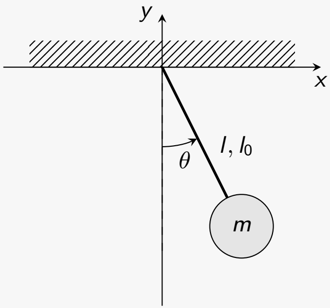
$$\dot{C} = \dot{l}, \qquad \ddot{C} = \ddot{l}$$

Index-1 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ \ddot{l} \end{bmatrix}$$

$$C(t=0) = 0, \qquad \dot{C}(t=0) = 0$$

Example: Constrained Pendulum



Index-3 DAE

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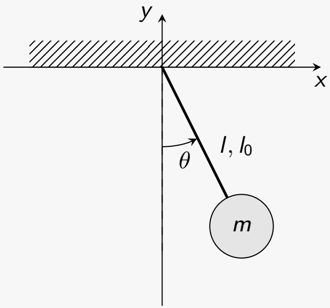
$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ \ddot{l} \end{bmatrix}$$

$$C(t=0) = 0, \qquad \dot{C}(t=0) = 0$$

ODE

$$\begin{bmatrix} \ddot{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} \frac{g}{l}\sin\theta \\ ml\dot{\theta}^2 + mg\cos\theta \end{bmatrix}, \quad l = l_0$$

Example: Constrained Pendulum



$\lambda =$ tension in the rod

Index-3 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ l - l_0 \end{bmatrix}$$

Index reduction:

$\dot{C} = \dot{l}, \qquad \ddot{C} = \ddot{l}$

Index-1 DAE

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{\lambda}{m} \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \\ \ddot{l} \end{bmatrix}$$

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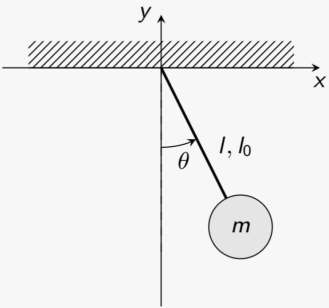
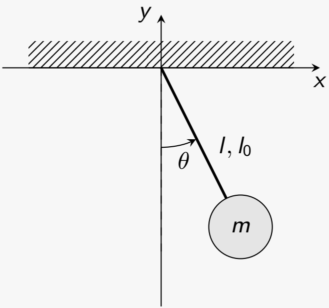
Example: Pendulum in Cartesian Coordinates



$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

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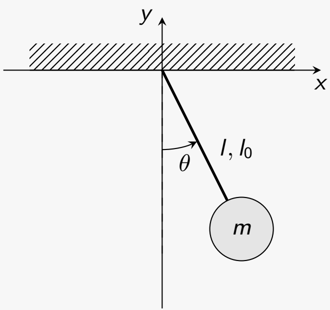
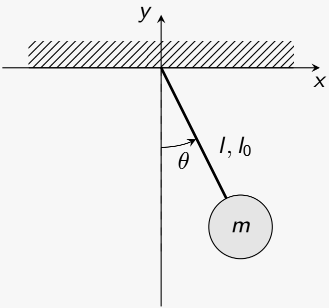
Example: Pendulum in Cartesian Coordinates



$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



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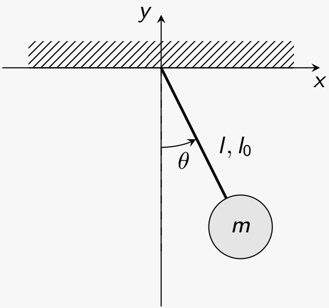
Example: Pendulum in Cartesian Coordinates



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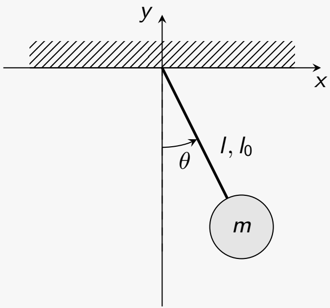
$$T = \frac{1}{2} m \dot{p}^\top \dot{p} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2), \quad U = mgy$$

Example: Pendulum in Cartesian Coordinates



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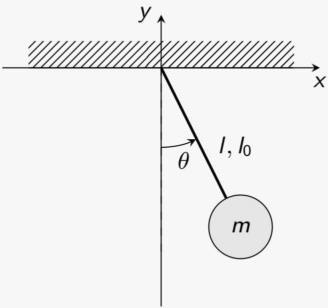
Example: Pendulum in Cartesian Coordinates



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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \end{bmatrix}$$

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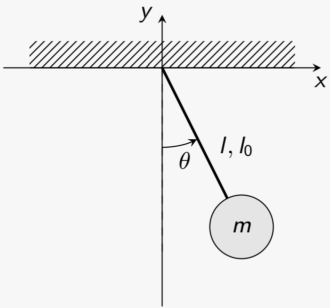

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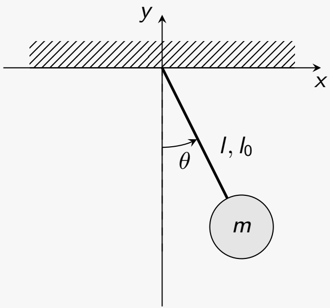

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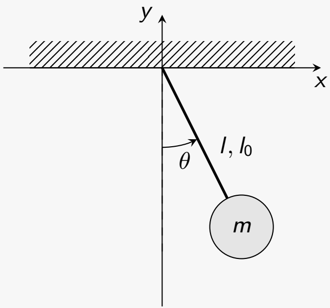
$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



Index-3 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

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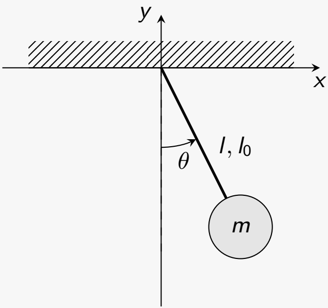

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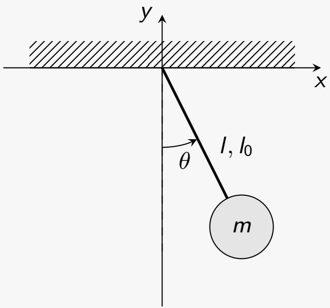
Index reduction:

Example: Pendulum in Cartesian Coordinates



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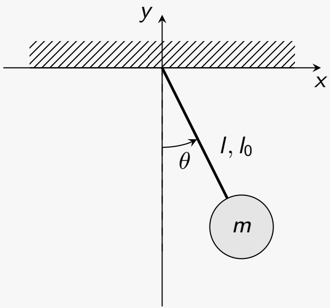
Index reduction:

$\dot{C} = \dot{x}x + \dot{y}y,$

Example: Pendulum in Cartesian Coordinates



Index-3 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

Index reduction:

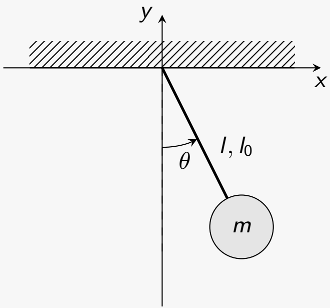
$$\dot{C} = \dot{x}x + \dot{y}y, \qquad \ddot{C} = \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2$$

Index-1 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

$$C(t = 0) = 0, \qquad \dot{C}(t = 0) = 0$$

Example: Pendulum in Cartesian Coordinates



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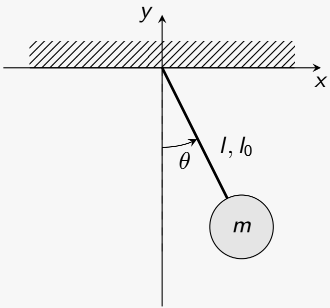
$$C(t = 0) = 0, \qquad \dot{C}(t = 0) = 0$$

Semi-implicit form

$$\begin{bmatrix} m & 0 & x \\ 0 & m & y \\ x & y & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ mg \\ \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates



$$\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \text{rod tension}$$

Index-3 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \frac{1}{2}(x^2 + y^2 - l_0^2) \end{bmatrix}$$

Index reduction:

$$\dot{C} = \dot{x}x + \dot{y}y, \qquad \ddot{C} = \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2$$

Index-1 DAE

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\lambda x}{m} \\ -g - \frac{\lambda y}{m} \\ \ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 \end{bmatrix}$$

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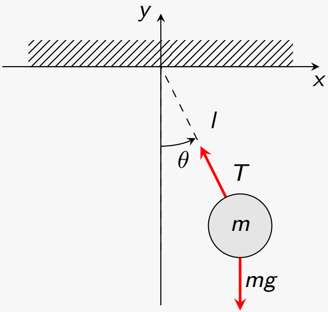
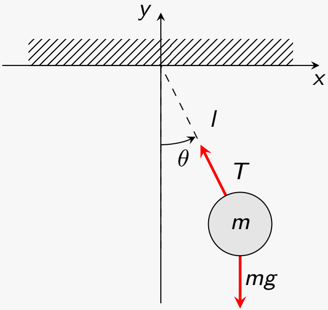
Example: Pendulum in Cartesian Coordinates



Newton's Approach

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}T^x\\ -mg + T^y\end{bmatrix}$$

Example: Pendulum in Cartesian Coordinates

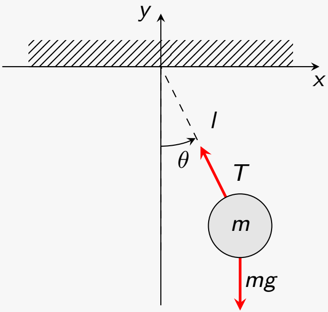


Newton's Approach

$$m\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} T^x \\ -mg + T^y \end{bmatrix}$$

Rod tension:

Example: Pendulum in Cartesian Coordinates



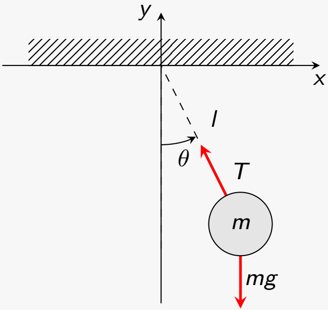
Newton's Approach

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} T^x \\ -mg + T^y \end{bmatrix}$$

Rod tension:

$$T_{\mathrm{g}} = mg \cos\theta = mg \frac{y}{\sqrt{x^2 + y^2}} = mg \frac{y}{l}$$

Example: Pendulum in Cartesian Coordinates



Newton's Approach

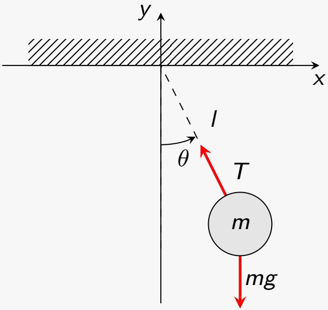
$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} T^x \\ -mg + T^y \end{bmatrix}$$

Rod tension:

$$T_{\mathrm{g}} = mg \cos\theta = mg \frac{y}{\sqrt{x^2 + y^2}} = mg \frac{y}{l}$$

$$T_{\mathrm{c}} = ml\dot{\theta} = m \frac{\dot{x}^2 + \dot{y}^2}{l}$$

Example: Pendulum in Cartesian Coordinates



Newton's Approach

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}T^x\\ -mg + T^y\end{bmatrix}$$

Rod tension:

$$T_{\mathrm{g}} = mg\cos\theta = mg\frac{y}{\sqrt{x^2+y^2}} = mg\frac{y}{l}$$

$$T_{\mathrm{c}} = ml\dot{\theta} = m\frac{\dot{x}^2+\dot{y}^2}{l}$$

In the x and y directions:

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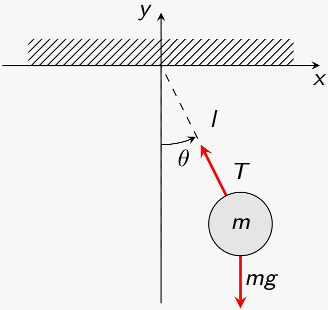
Example: Pendulum in Cartesian Coordinates



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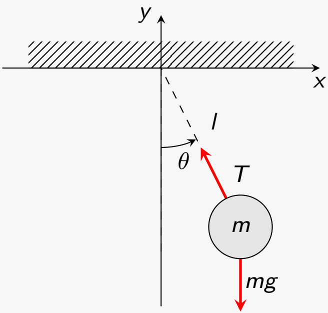

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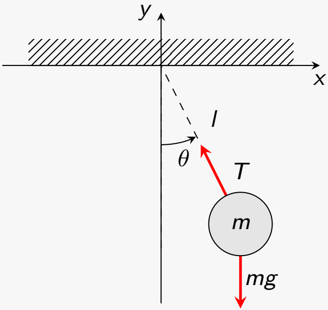

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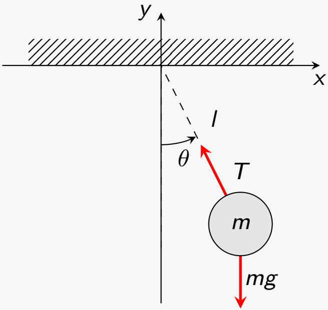

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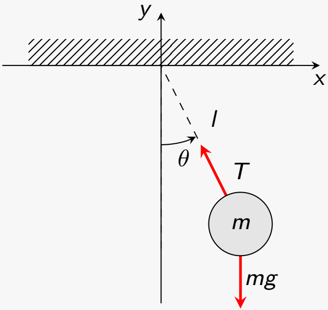

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$$T^x = \lambda x, \qquad T^y = \lambda y$$

Principle of Virtual Works

How to include external forces/torques?

- Generalised coordinates $\boldsymbol{q} \Rightarrow$ generalised forces F^q

Typically, it is natural to express forces F in a coordinate frame $x \neq \boldsymbol{q}$.

How can we compute F^q?

- Virtual displacement: δx
- Principle of Virtual Works: $\delta W := F\delta x = F^q \delta \boldsymbol{q}$
- $x = x(\boldsymbol{q})$, then: $\delta x = \frac{\mathrm{d}x}{\mathrm{d}\boldsymbol{q}} \delta \boldsymbol{q}$

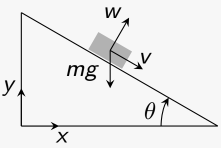
Generalised forces:

$$F^q = F \frac{\mathrm{d}x}{\mathrm{d}\boldsymbol{q}}$$

Examples

Examples

Mass on a frictionless inclined plane



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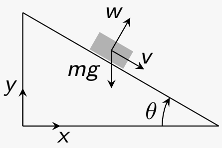

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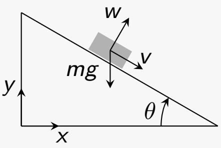

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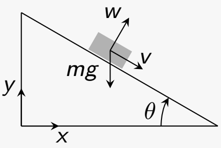

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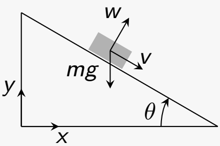

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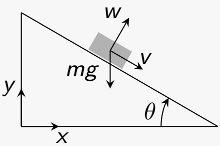

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The generalised force F^q is constant!

Examples

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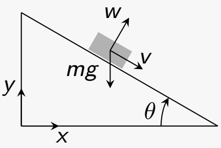

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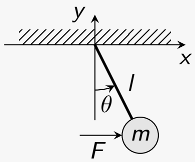
Pendulum with Horizontal Force



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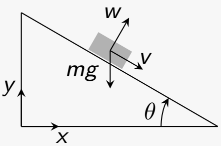

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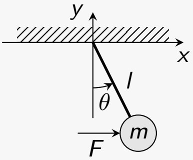

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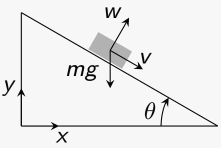

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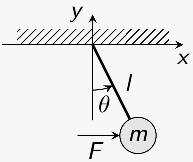

$$q = \theta, \qquad x = l\sin\theta,$$

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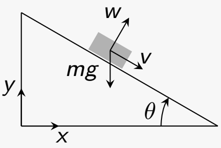

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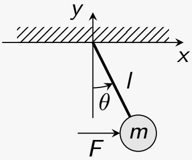

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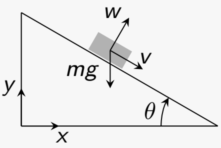

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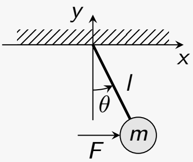
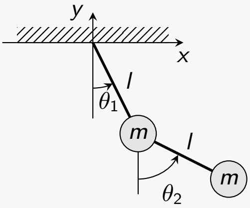


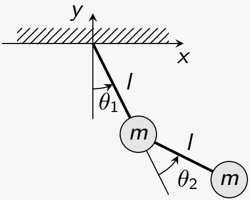
$$q = \theta, \qquad x = l\sin\theta, \qquad F^q = F\frac{\mathrm{d}x}{\mathrm{d}\theta} = Fl\cos\theta$$

The generalised force F^q depends on q!

Exercises

- Model a double pendulum in polar and cartesian coordinates (assume $l_1 = l_2 = l$, $m_1 = m_2 = m$)





- Model a pendulum attached to a crane in polar and cartesian coordinates
- Assume to be able to change the length of the crane rod