Aerospace Modeling Tutorial
Lecture 1 – Rigid Body Dynamics

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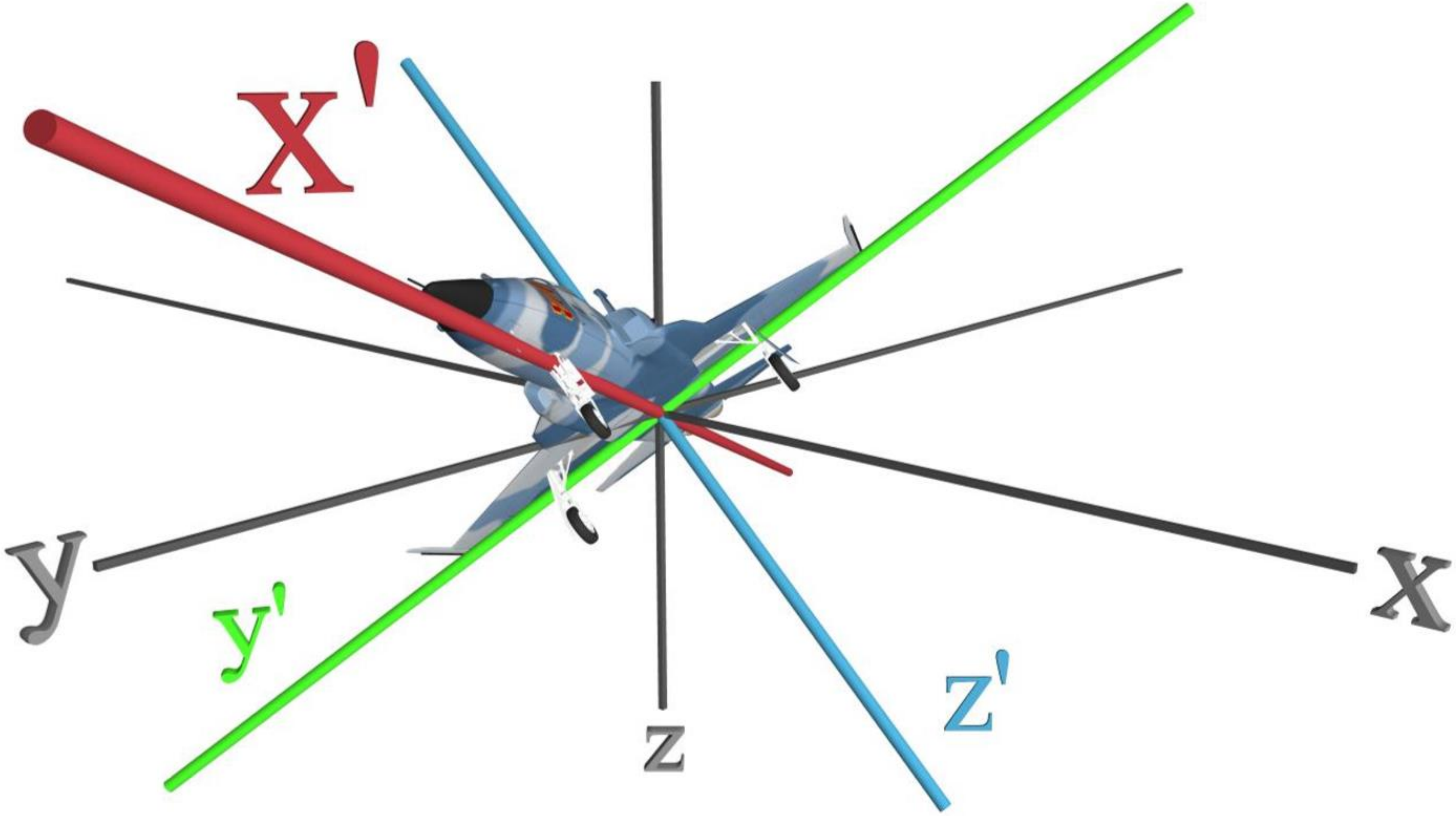
Reference frames



North-East-Down: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Body: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

Relative rotation: $\vec{\omega}$

Translational Dynamics

$$\dot{\vec{v}}_n = \frac{\vec{F}_n}{m} \qquad \dot{\vec{v}}_b = \frac{\vec{F}_b}{m} - \overrightarrow{\omega} \times \vec{v}_b$$

$$\dot{\vec{p}}_n = \vec{v}_n \qquad \dot{\vec{p}}_n = R^{\mathrm{T}} \vec{v}_b$$

Rotational Dynamics

$$T = J\dot{\omega} = J\ddot{\theta} \qquad \text{(1 dimensional)}$$

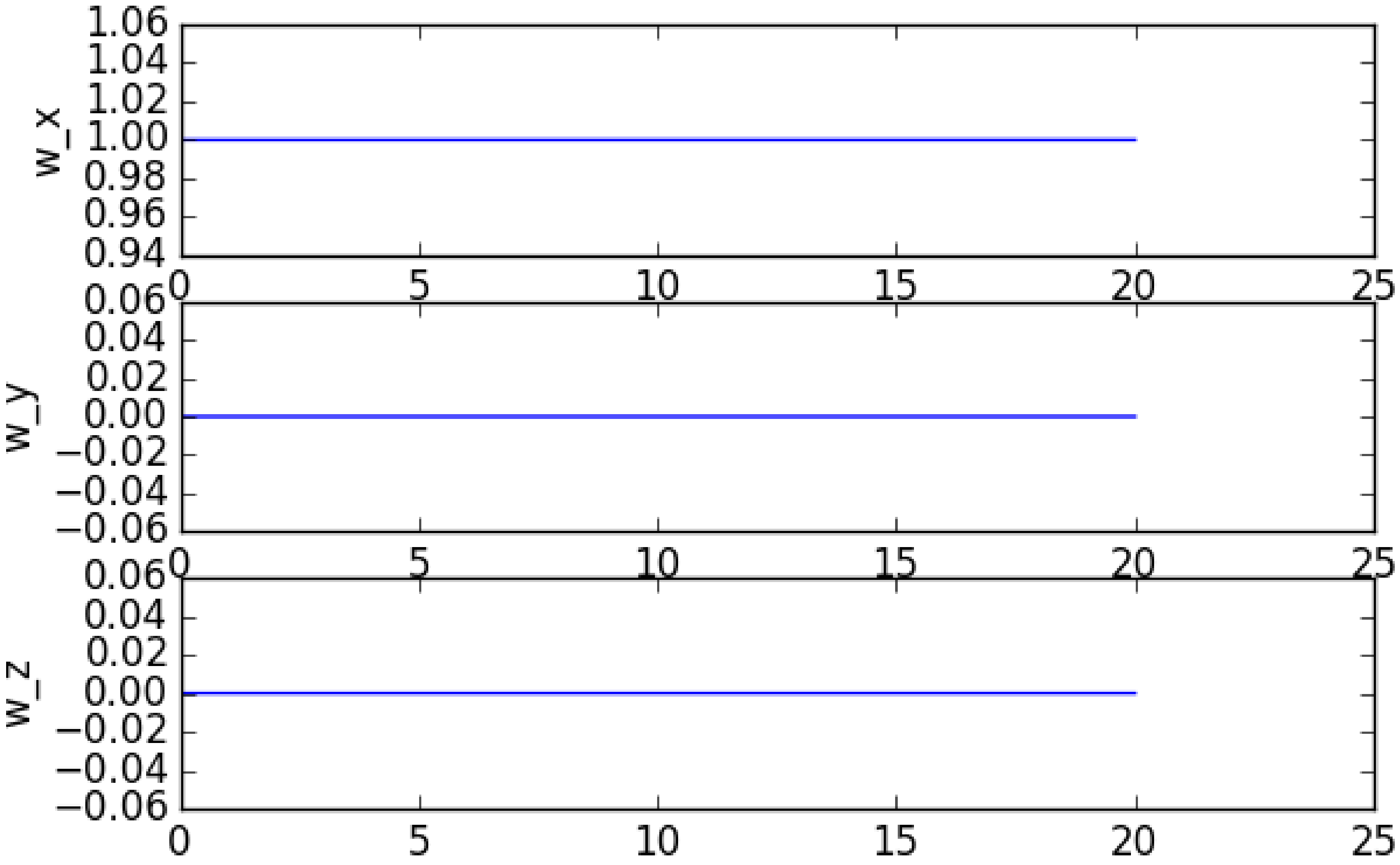
$$\bar{\bar{J}} \cdot \dot{\vec{\omega}} = \vec{\omega} \times \left(\bar{\bar{J}} \cdot \vec{\omega}\right) + \vec{T}_b \qquad \text{(3 dimensional)}$$

$$\dot{\vec{\omega}} = \bar{\bar{J}}^{-1}[\vec{\omega} \times \left(\bar{\bar{J}} \cdot \vec{\omega}\right) + \vec{T}_b]$$

Rotational sim

$$\omega(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

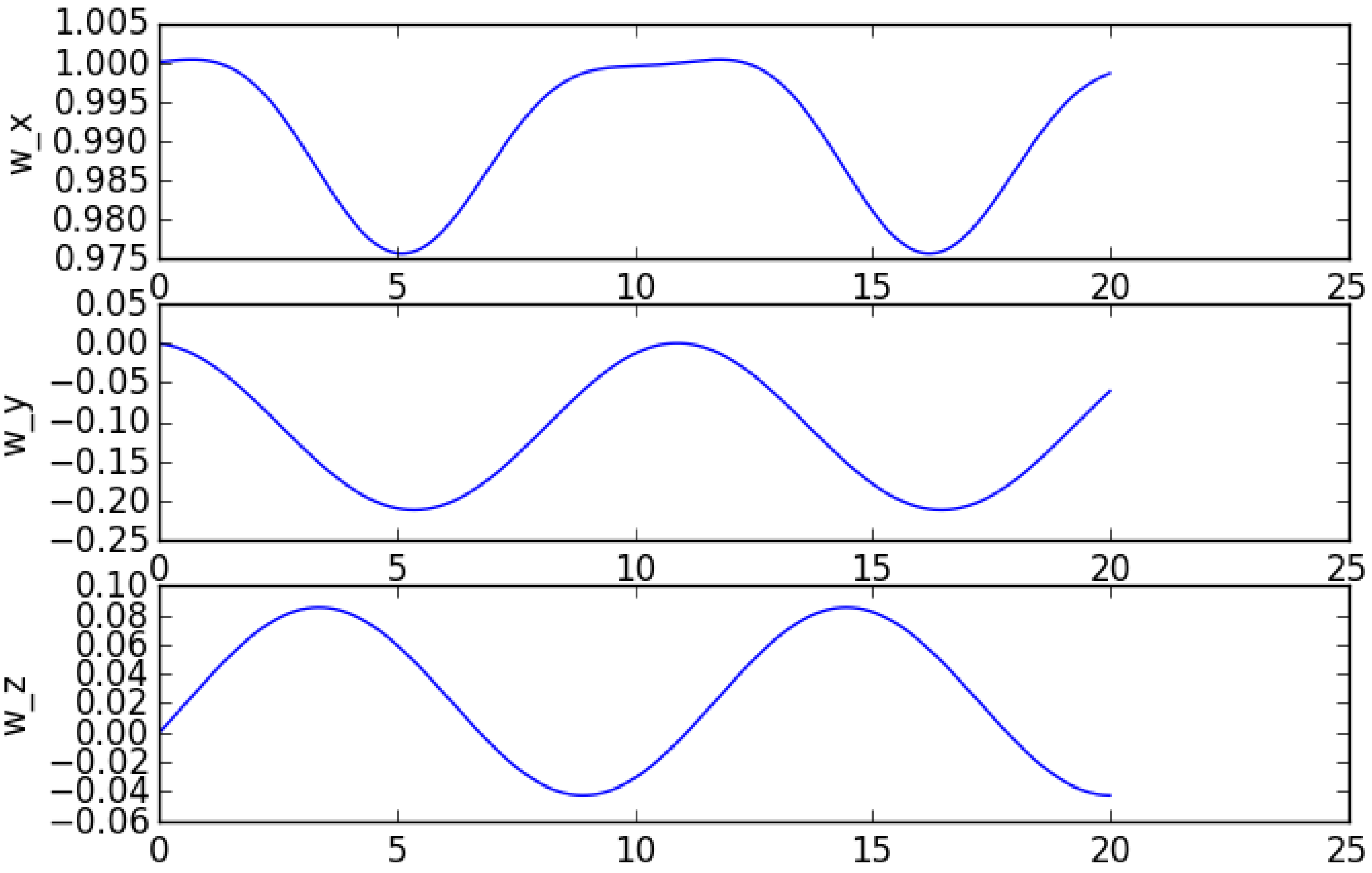
$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$



Rotational sim

$$\omega(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0.1 & 0 \\ 0.1 & 2 & 0.4 \\ 0 & 0.4 & 3 \end{pmatrix}$$



We know the angular velocity, but not the angle

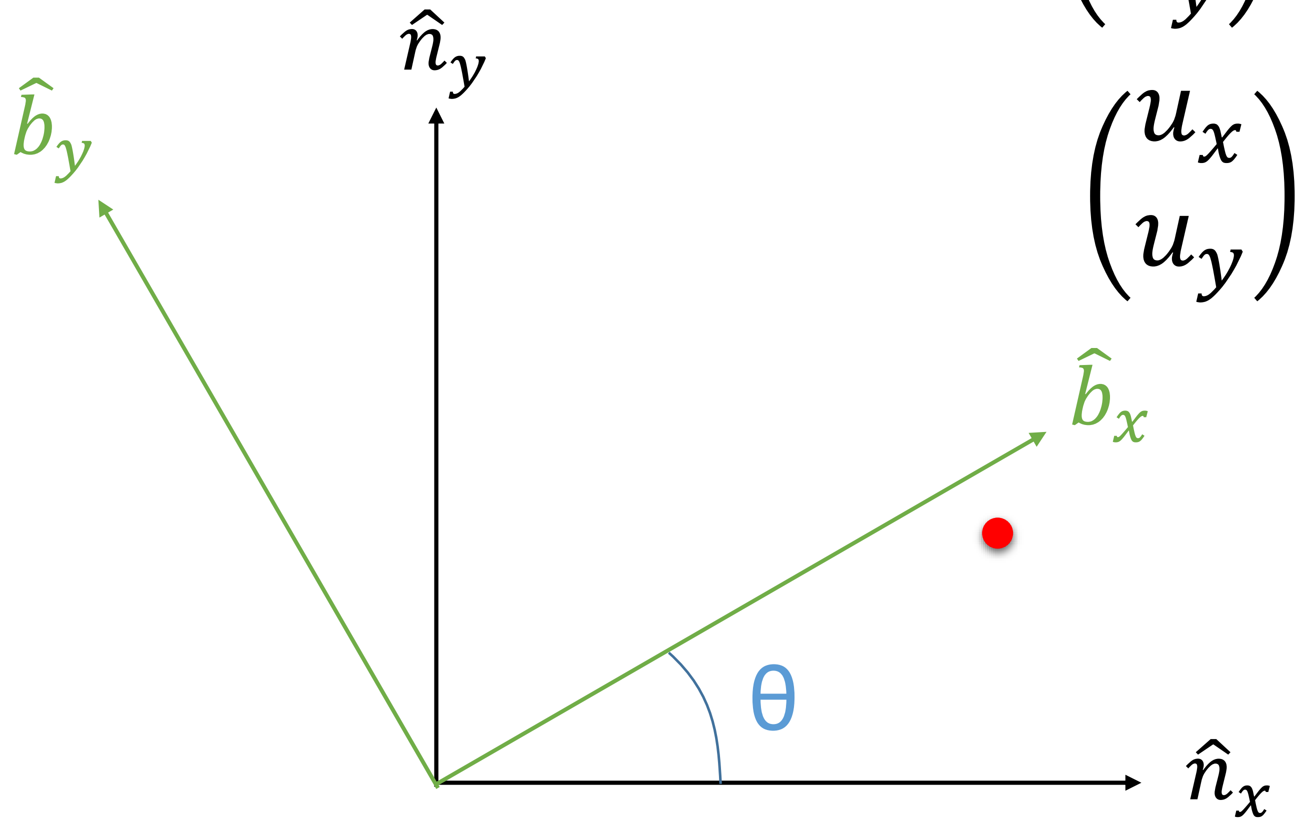
super easy 1 dimension

$$\dot{\omega} = T/J$$
$$\dot{\theta} = \omega$$

How to model rotations – 1 dimension

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 3.1 \\ -0.4 \end{pmatrix}$$



$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

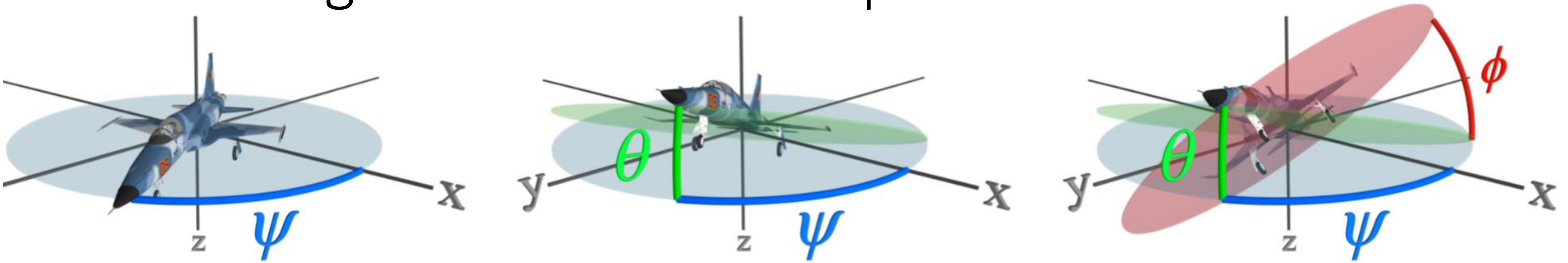
How to model rotations – 3 dimensional

First Idea:

Euler angles

(yaw, pitch, roll)

Euler angles – how do we express the rotation?



$$\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_e \\ v_e \\ w_e \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$$

$$\begin{pmatrix} u_b \\ v_b \\ w_b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\begin{pmatrix} u_b \\ v_b \\ w_b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_e \\ v_e \\ w_e \end{pmatrix}$$

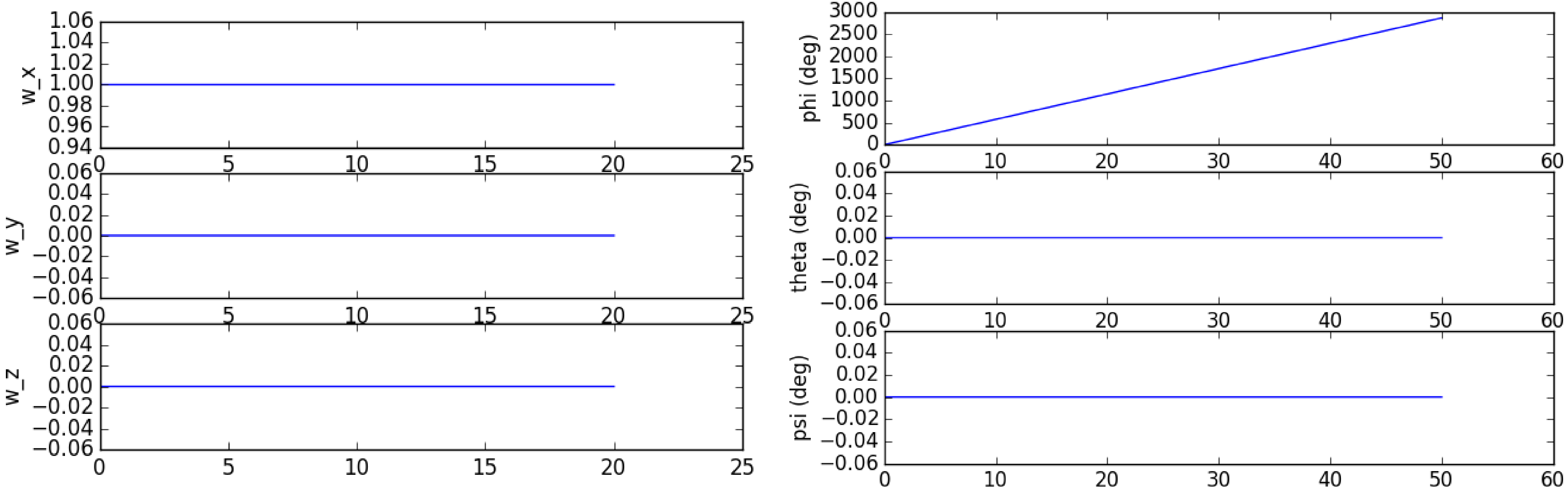
$$\begin{pmatrix} u_b \\ v_b \\ w_b \end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \cos(\psi)\sin(\theta)\sin(\phi) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\theta)\sin(\phi)\sin(\psi) & \cos(\theta)\sin(\phi) \\ \cos(\phi)\cos(\psi)\sin(\theta) + \sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\phi) + \cos(\phi)\sin(\theta)\sin(\psi) & \cos(\theta)\cos(\phi) \end{pmatrix} \begin{pmatrix} u_e \\ v_e \\ w_e \end{pmatrix}$$

How do we use this with our rigid body equations?

$$\dot{\vec{\omega}} = \bar{\bar{J}}^{-1}[\vec{\omega} \times (\bar{\bar{J}} \cdot \vec{\omega}) + \vec{T}]$$

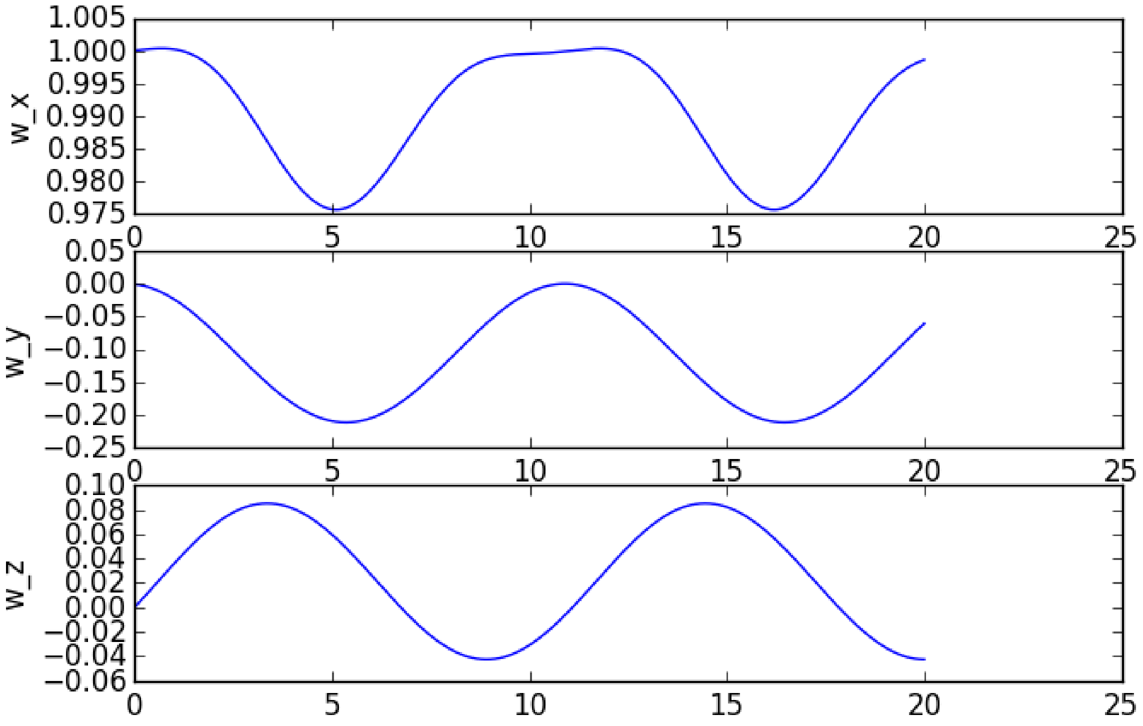
$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = ?$$

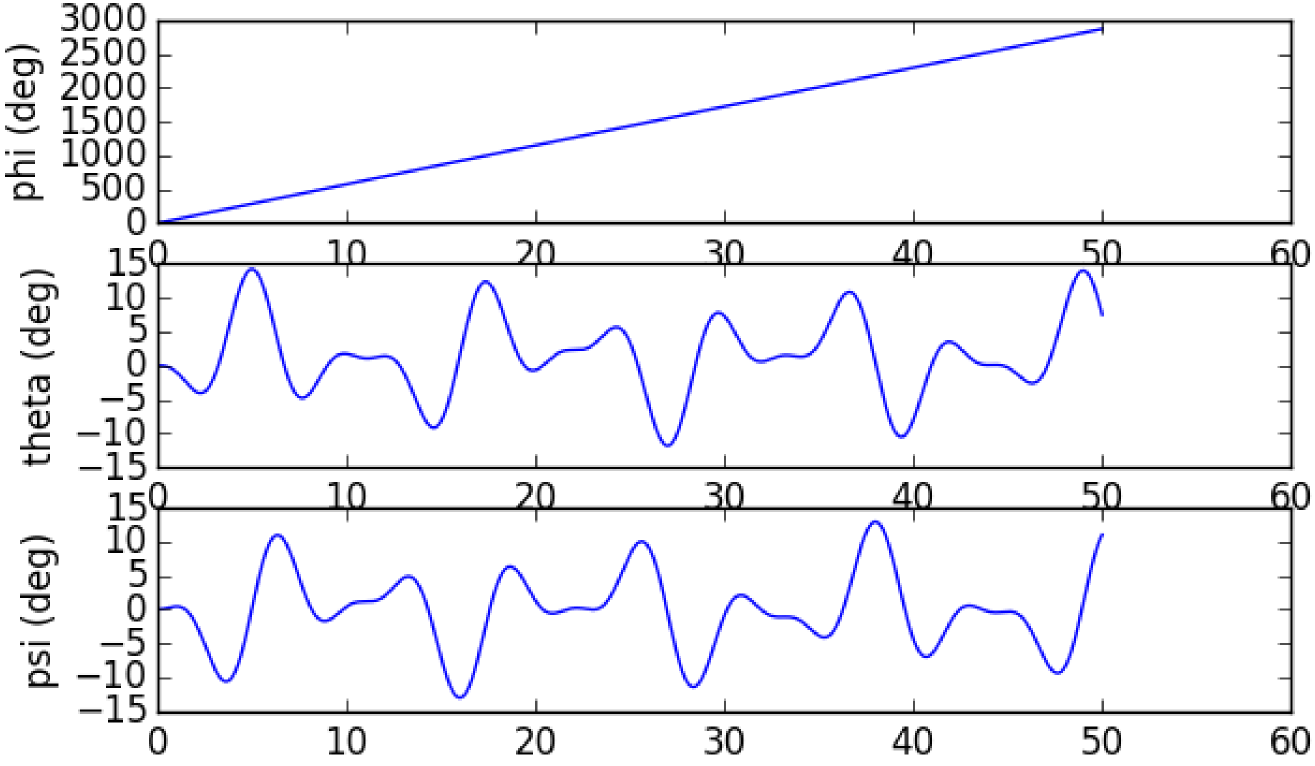
How do we use this with our rigid body equations?



(Sorry for different time scales)

How do we use this with our rigid body equations?





(Sorry for different time scales)

Euler angles – why don't we use them?

$$\begin{pmatrix} u_b \\ v_b \\ w_b \end{pmatrix} = \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \cos(\psi)\sin(\theta)\sin(\phi) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\theta)\sin(\phi)\sin(\psi) & \cos(\theta)\sin(\phi) \\ \cos(\phi)\cos(\psi)\sin(\theta) + \sin(\phi)\sin(\psi) & -\cos(\psi)\sin(\phi) + \cos(\phi)\sin(\theta)\sin(\psi) & \cos(\theta)\cos(\phi) \end{pmatrix} \begin{pmatrix} u_e \\ v_e \\ w_e \end{pmatrix}$$

These are very nonlinear

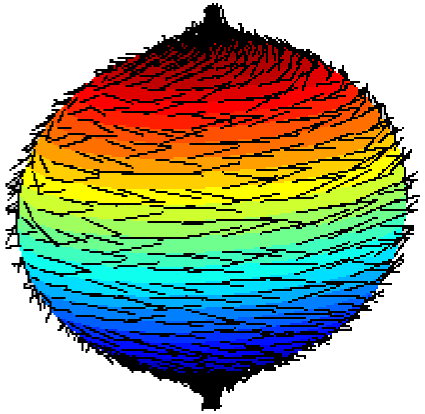
$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Major problem: **Harry Ball Theorem**

(Every cow must have at least one cowlick)

(You can't comb the hair on a coconut)

$$\lim_{\theta \to \frac{\pi}{2}} T(\phi, \theta, \psi) = \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi - \psi) & cos(\phi - \psi) & 0 \\ \cos(\phi - \psi) & -sin(\phi - \psi) & 0 \end{pmatrix}$$



Q: What do we use instead of Euler Angles?

A: Quaternions or Rotation Matrices!

Quaternions in 15 seconds

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$$\vec{v}_b = \begin{pmatrix} (2q_0^2 - 1) + 2q_1^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & (2q_0^2 - 1) + 2q_2^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & (2q_0^2 - 1) + 2q_3^2 \end{pmatrix} \vec{v}_e$$

$$\frac{d}{dt}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\vec{\omega}^T \\ \vec{\omega} & -\vec{\omega}\times \end{pmatrix}\begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix}$$

Very compact and elegant representation of attitude …which we will not discuss today

Rotation Matrices a.k.a. Direction Cosine Matrices

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

How to rotate vectors from one frame to another?

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Convention: $\vec{v}_b = R\vec{v}_n$

Word to the wise:
Everyone uses different conventions.
Stick to one. Always check other people's convention.

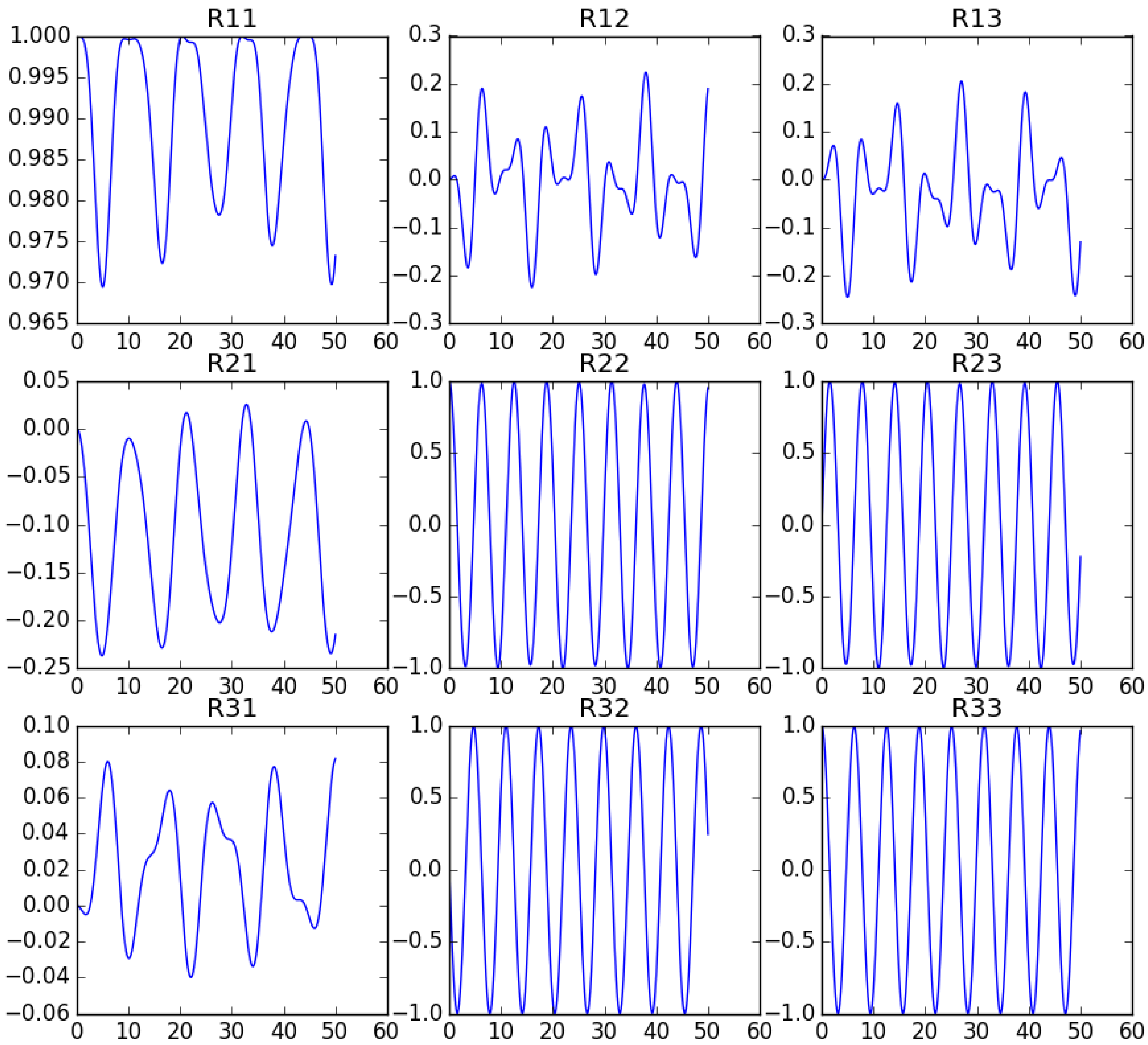
Rotation Matrices a.k.a. Direction Cosine Matrices

Derivative with respect to ω

$$\begin{pmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{pmatrix} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

DCM sim

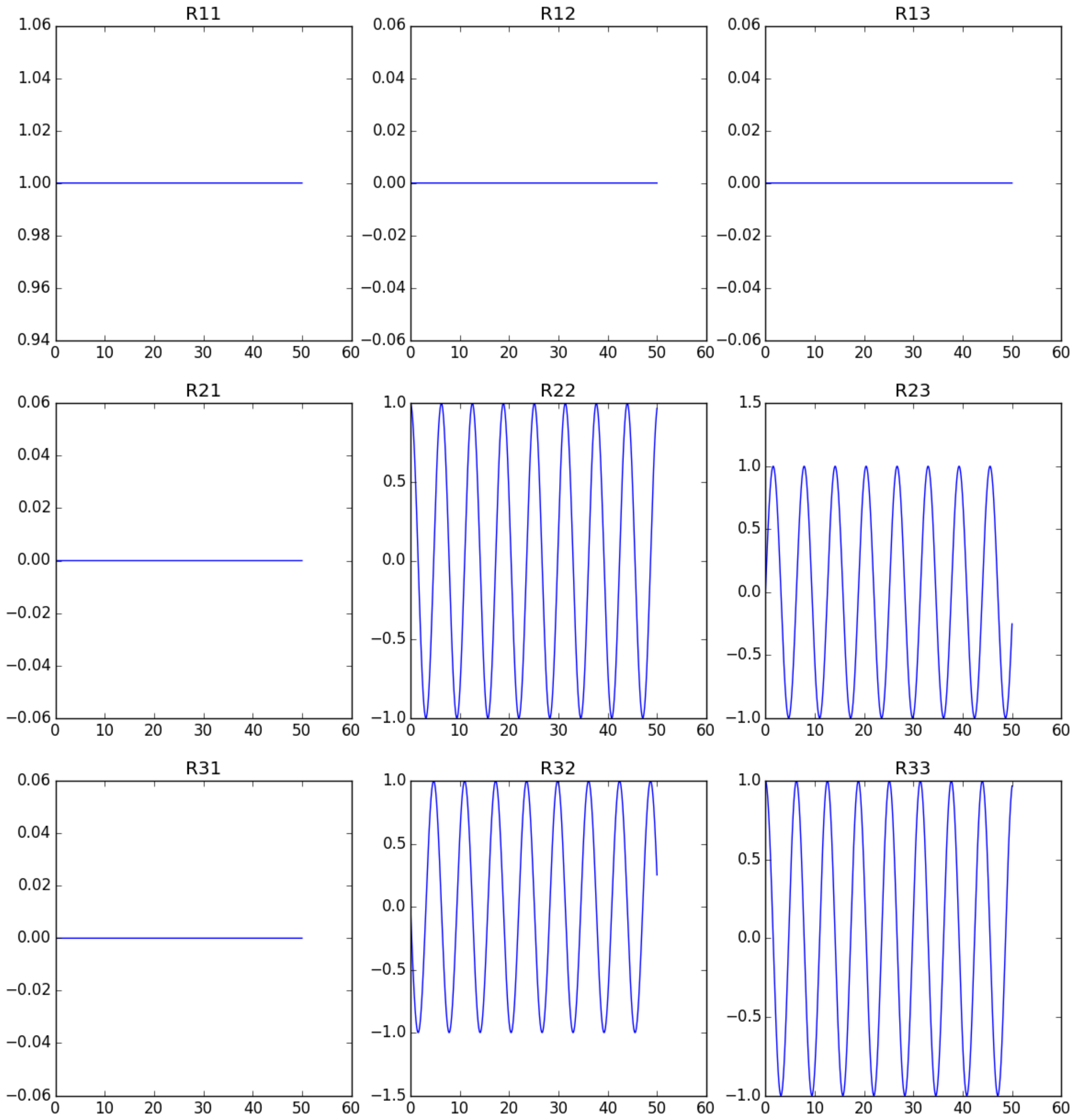
$$J = \begin{pmatrix} 1 & 0.1 & 0 \\ 0.1 & 2 & 0.4 \\ 0 & 0.4 & 3 \end{pmatrix}$$



DCM sim

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} u_b \\ v_b \\ w_b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}$$



What makes DCMs hard?

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} e_x^{\mathrm{T}} \\ e_y^{\mathrm{T}} \\ e_z^{\mathrm{T}} \end{pmatrix}$$

Hard to visualize

- Solution: convert to Euler angles before plotting

Must be initialized right-handed and orthonormal

- Easiest solution: initialize to identity
- Easy solution: initialize as Euler angles then convert to DCM.
- If initial angle is free/unknown, use hard solution: enforce orthonormality as a constraint

$$e_x^{\mathrm{T}} e_x = 1$$
$$e_x^{\mathrm{T}} e_y = 0$$
$$e_y^{\mathrm{T}} e_y = 1$$

$$e_x \times e_y - e_z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Only enforce this at one time point!
Doing this more than once destroys LICQ!

$$R_1 = R_2$$

Matching conditions have 9 equations and 3 degrees of freedom

- Solution: Enforce small relative rotation = 0

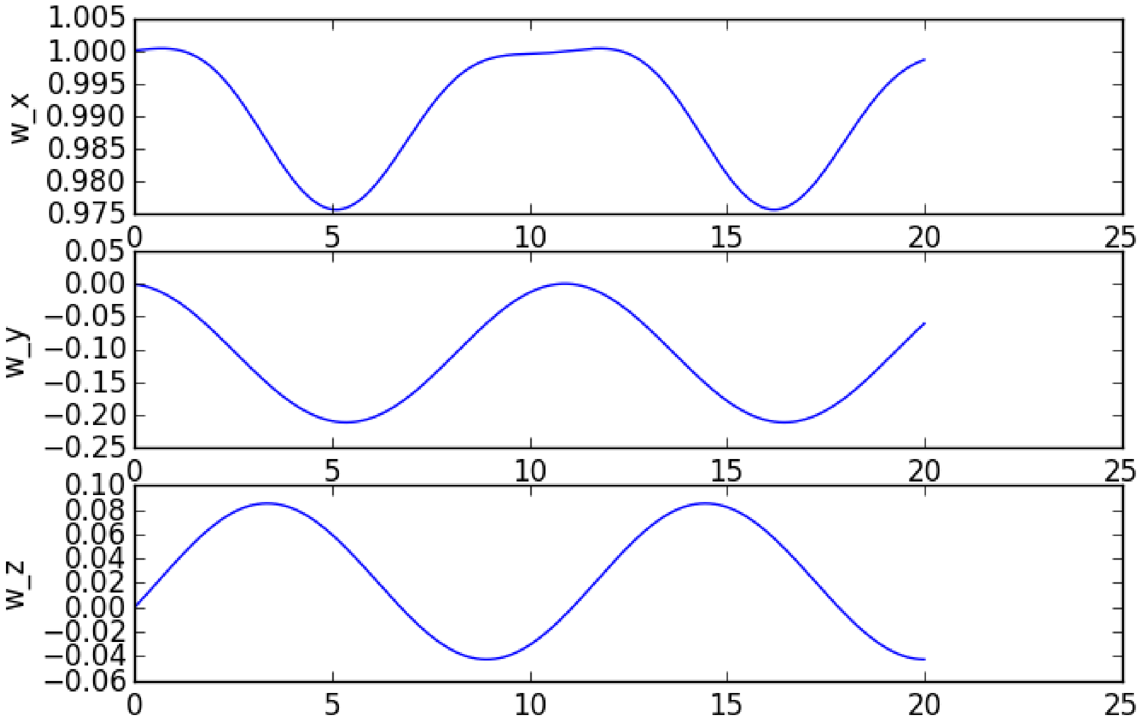
$$R_1^{\mathrm{T}} R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

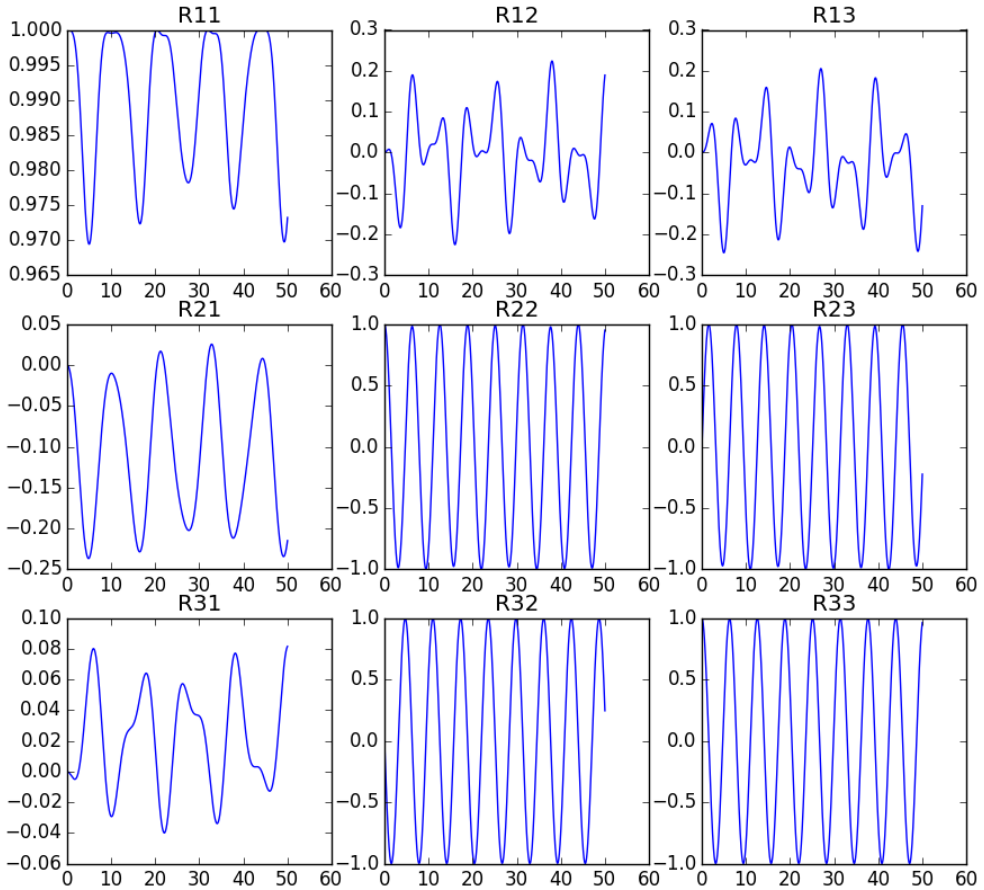
Only enforce these three

IMPORTANT:
Also enforce diagonal elements positive

Homework 1: Reproduce these plots

$$\omega(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 & 0.1 & 0 \\ 0.1 & 2 & 0.4 \\ 0 & 0.4 & 3 \end{pmatrix}$$





(Sorry for different time scales)

Homework 2:
Match two simulations with different states

$$\vec{x}_1 = \begin{pmatrix} \vec{p}_n \\ \vec{v}_n \\ \vec{\omega} \\ R \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} \vec{p}_n \\ \vec{v}_b \\ \vec{\omega} \\ R \end{pmatrix}$$

$$\vec{F}_n(t) = \begin{pmatrix} 0.3t + 0.1 \sin 3t \\ 0.4t + 0.2 \sin 4t \\ 0.5t + 0.1 \sin 5t \end{pmatrix}$$

$$p(0) = v(0) = \omega(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{T}_b(t) = \begin{pmatrix} 1.5 \sin 2t \\ 2 \sin 1t \\ \sin 0.5t \end{pmatrix}$$

$$R(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0.1 & 0.3 \\ 0.1 & 2 & 0.2 \\ 0.3 & 0.2 & 3 \end{pmatrix}$$

Homework 3:
minimum torque satellite de-tumble
Multiple shooting with 200 timesteps, rk4 integrator

Objective=$\sum_k \vec{T}_k^{\mathrm{T}} \vec{T}_k$

$$R(0) = \begin{pmatrix} 0.07 & 0.46 & 0.88 \\ -0.89 & -0.37 & 0.26 \\ 0.45 & -0.80 & 0.39 \end{pmatrix} \qquad \omega(0) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0.1 & 0.3 \\ 0.1 & 2 & 0.2 \\ 0.3 & 0.2 & 3 \end{pmatrix} \qquad R(6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \omega(6) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Homework 4: OPTIONAL BONUS QUESTION

same problem as homework 3, but in minimal time

Objective is now end time

Add bounds on the control: $\begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} \leq \vec{T}_k \leq \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$