Direct Single and Direct Multiple Shooting

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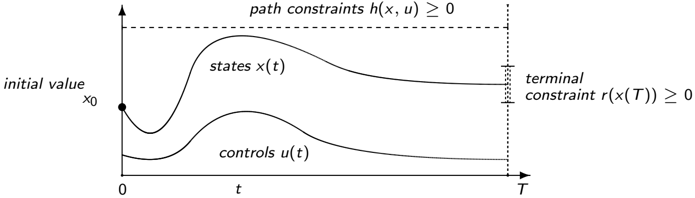
Overview

- Direct Single Shooting
- The Gauss-Newton Method
- Direct Multiple Shooting
- Structure Exploitation by Condensing
- Structure Exploitation by Riccati Recursion

Simplified Optimal Control Problem in ODE

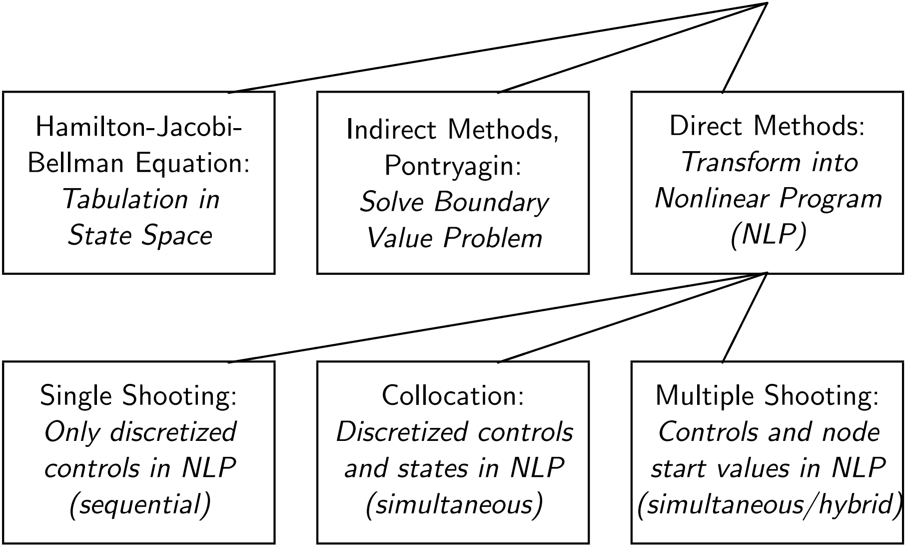


$$\underset{x(\cdot),\, u(\cdot)}{\text{minimize}} \quad \int_0^T L(x(t), u(t))\, dt \; + \; E\,(x(T))$$

subject to

$$\begin{aligned}
x(0) - x_0 &= 0, & & \text{(fixed initial value)} \\
\dot{x}(t) - f(x(t), u(t)) &= 0, & t \in [0, T], & \text{(ODE model)} \\
h(x(t), u(t)) &\geq 0, & t \in [0, T], & \text{(path constraints)} \\
r\,(x(T)) &\geq 0 & & \text{(terminal constraints).}
\end{aligned}$$

Recall: Optimal Control Family Tree

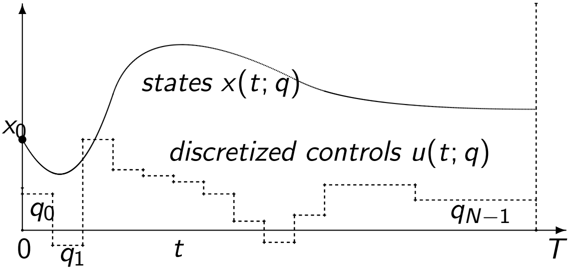


Direct Methods

- "First discretize, then optimize"
- Transcribe infinite problem into finite dimensional, **Nonlinear Programming Problem (NLP)**, and solve NLP.
- Pros and Cons:
  - \+ Can use state-of-the-art methods for NLP solution.
  - \+ Can treat inequality constraints and multipoint constraints much easier.
  - \- Obtains only suboptimal/approximate solution.
- Nowadays most commonly used methods due to their easy applicability and robustness.

Direct Single Shooting [Hicks, Ray 1971; Sargent, Sullivan 1977]

Discretize controls $u(t)$ on fixed grid $0 = t_0 < t_1 < \ldots < t_N = T$, regard states $x(t)$ on $[0, T]$ as dependent variables.



Use numerical integration to obtain state as function $x(t; q)$ of finitely many control parameters $q = (q_0, q_1, \ldots, q_{N-1})$

NLP in Direct Single Shooting

After control discretization and numerical ODE solution, obtain NLP:

$$\underset{q}{\text{minimize}} \quad \int_0^T L(x(t;q), u(t;q))\, dt + E\,(x(T;q))$$

subject to

$$h(x(t_i;q), u(t_i;q)) \geq 0, \quad i = 0, \ldots, N-1, \qquad \textit{(discretized path constraints)}$$

$$r\,(x(T;q)) \geq 0. \qquad \textit{(terminal constraints)}$$

Solve with finite dimensional optimization solver, e.g. Sequential Quadratic Programming (SQP).

Solution by Standard SQP

Summarize problem as

$$\min_q F(q) \ \text{ s.t. } \ H(q) \geq 0.$$

Solve e.g. by Sequential Quadratic Programming (SQP), starting with guess q^0 for controls. $k := 0$

1. Evaluate $F(q^k), H(q^k)$ by ODE solution, and derivatives!
2. Compute correction Δq^k by solution of QP:

$$\min_{\Delta q} \nabla F(q_k)^\top \Delta q + \frac{1}{2} \Delta q^\top A^k \Delta q \ \text{ s.t. } \ H(q^k) + \nabla H(q^k)^\top \Delta q \geq 0.$$

3. Perform step $q^{k+1} = q^k + \alpha_k \Delta q^k$ with step length α_k determined by line search.

Hessian in Quadratic Subproblem

Matrix A^k in QP

$$\min_{\Delta q} \nabla F(q_k)^\top \Delta q + \frac{1}{2}\Delta q^\top A^k \Delta q \quad \text{s.t.} \quad H(q^k) + \nabla H(q^k)^\top \Delta q \geq 0.$$

is called the Hessian matrix. Several variants exist:

- exact Hessian: $A^k = \nabla_q^2 \mathcal{L}(q, \mu)$ with μ the constraint multipliers. Delivers fast quadratic local convergence.
- Update Hessian using consecutive Lagrange gradients, e.g. by BFGS formula: superlinear
- In case of least squares objective $F(q) = \frac{1}{2}\|R(q)\|_2^2$ can also use Gauss-Newton Hessian (good linear convergence).

$$A^k = \left(\frac{\partial R}{\partial q}(q^k)\right)^\top \frac{\partial R}{\partial q}(q^k)$$

The Generalized Gauss-Newton Method

- Aim: solve constrained nonlinear least squares problems:

$$\min_q \frac{1}{2}\|R(q)\|_2^2 \quad \text{s.t.} \quad H(q) \geq 0.$$

- Generalized Gauss-Newton solves in each iteration:

$$\min_{\Delta q} \frac{1}{2}\|R(q_k)+\nabla R(q_k)^\top \Delta q\|_2^2 \quad \text{s.t.} \quad H(q^k)+\nabla H(q^k)^\top \Delta q \geq 0.$$

- This is a QP and equivalent to

$$\min_{\Delta q} \quad \underbrace{R(q_k)^\top \nabla R(q_k)^\top}_{=\nabla F(q_k)^\top} \Delta q + \frac{1}{2}\Delta q^\top \underbrace{\nabla R(q_k)\nabla R(q_k)^\top}_{=:A_k} \Delta q$$

$$\text{s.t.} \quad H(q^k) + \nabla H(q^k)^\top \Delta q \geq 0.$$

Properties of Gauss-Newton Hessian

- Gauss-Newton Hessian $A_k := \nabla R(q_k) \nabla R(q_k)^\top$ is symmetric and has only non-zero eigenvalues. Thus, QP subproblems are convex.
- A_k is similar to $\nabla_q^2 \mathcal{L}(q_k, \mu_k)$, but not equal.
- Using $\mathcal{L}(q, \mu) = \frac{1}{2}\|R(q)\|_2^2 - H(q)^\top \mu$ and

$$\nabla^2 \left(\frac{1}{2} \|R(q)\|_2^2 \right) = \nabla R(q_k) \nabla R(q_k)^\top + \sum_{i=1}^{n_R} R_i(q) \nabla^2 R_i(q)$$

we get $\nabla_q^2 \mathcal{L}(q, \mu) =$

$$\nabla R(q) \nabla R(q)^\top + \underbrace{\sum_{i=1}^{n_R} R_i(q) \nabla^2 R_i(q) - \sum_{i=1}^{n_H} \mu_i \nabla^2 H_i(q)}_{\text{error (small if } \|R(q)\| \text{ small at solution)}}$$

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- The Gauss-Newton Method
- **Direct Multiple Shooting**
- Structure Exploitation by Condensing
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Direct Multiple Shooting [Bock and Plitt, 1981]

- Discretize controls piecewise on a coarse grid

$$u(t) = q_i \quad \text{for} \quad t \in [t_i, t_{i+1}]$$

- Solve ODE on each interval $[t_i, t_{i+1}]$ numerically, starting with artificial initial value s_i:

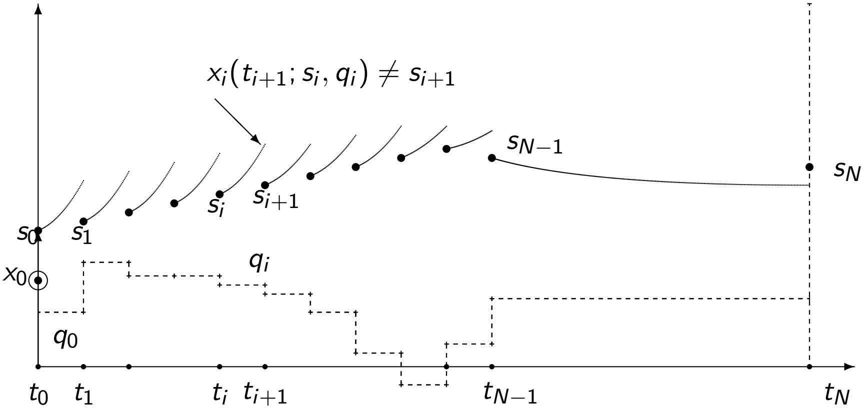
$$\begin{aligned} \dot{x}_i(t; s_i, q_i) &= f(x_i(t; s_i, q_i), q_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i; s_i, q_i) &= s_i. \end{aligned}$$

  Obtain trajectory pieces $x_i(t; s_i, q_i)$.

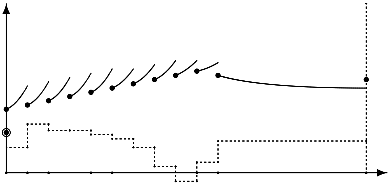
- Also numerically compute integrals

$$l_i(s_i, q_i) := \int_{t_i}^{t_{i+1}} L(x_i(t_i; s_i, q_i), q_i) dt$$

Sketch of Direct Multiple Shooting



NLP in Direct Multiple Shooting



$$\underset{s,q}{\text{minimize}} \quad \sum_{i=0}^{N-1} l_i(s_i, q_i) \; + \; E\left(s_N\right)$$

subject to

$$\begin{aligned} s_0 - x_0 &= 0, & &\text{(initial value)} \\ s_{i+1} - x_i(t_{i+1}; s_i, q_i) &= 0,\ i = 0, \ldots, N-1, & &\text{(continuity)} \\ h(s_i, q_i) &\geq 0,\ i = 0, \ldots, N, & &\text{(discretized path constraints} \\ r\left(s_N\right) &\geq 0. & &\text{(terminal constraints)} \end{aligned}$$

Structured NLP

- Summarize all variables as $w := (s_0, q_0, s_1, q_1, \dots, s_N)$.
- Obtain structured NLP

$$\min_{w} \ F(w) \quad \text{s.t.} \quad \begin{cases} G(w) = 0 \\ H(w) \geq 0. \end{cases}$$

- Jacobian $\nabla G(w^k)^\top$ contains dynamic model equations.
- Jacobians and Hessian of NLP are block sparse, can be exploited in numerical solution procedure.

QP = Discrete Time Problem

$$\min_{x,\,u} \sum_{i=0}^{N-1} \begin{bmatrix} 1 \\ \Delta s_i \\ \Delta q_i \end{bmatrix}^\top \begin{bmatrix} 0 & q_i^\top & s_i^\top \\ q_i & Q_i & S_i^\top \\ s_i & S_i & R_i \end{bmatrix} \begin{bmatrix} 1 \\ \Delta s_i \\ \Delta q_i \end{bmatrix} + \begin{bmatrix} 1 \\ \Delta s_N \end{bmatrix}^\top \begin{bmatrix} 0 & p_N^\top \\ p_N & P_N \end{bmatrix} \begin{bmatrix} 1 \\ \Delta s_N \end{bmatrix}$$

subject to

$$\begin{aligned} \Delta s_0 - x_0^{\mathrm{fix}} &= 0, && && \text{(initial)} \\ \Delta s_{i+1} - A_i \Delta s_i - B_i \Delta q_i - c_i &= 0, && i = 0, \dots, N-1, && \text{(system)} \\ C_i \Delta s_i + D_i \Delta q_i - c_i &\leq 0, && i = 0, \dots, N-1, && \text{(path)} \\ C_N \Delta s_N - c_N &\leq 0, && && \text{(terminal)} \end{aligned}$$

Interpretation of Continuity Conditions

- In direct multiple shooting, continuity conditions $s_{i+1} = x_i(t_{i+1}; s_i, q_i)$ represent discrete time dynamic system.
- *Linearized* reduced continuity conditions (used in *condensing* to eliminate $\Delta s_1, \ldots, \Delta s_N$) represent **linear discrete time system:**

$$\Delta s_{i+1} = (x_i(t_{i+1}; s_i, q_i) - s_{i+1}) + X_i \Delta s_i^x + Y_i \Delta q_i = 0, \qquad i = 0, \ldots, N-1.$$

- If original system is linear, continuity is perfectly satisfied in all SQP iterations.
- Lagrange multipliers λ_i for the continuity conditions are approximation of **adjoint variables**. They indicate the costs of continuity.

Condensing Technique [Bock, Plitt, 1984]

As before in multiple shooting for BVPs, can use "condensing" of linear system equations

$$\begin{bmatrix} X_0 & Y_0 & -\mathbb{I} & & & & & \\ & & X_1 & Y_1 & -\mathbb{I} & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & X_{N-1} & Y_{N-1} & -\mathbb{I} \end{bmatrix} \begin{bmatrix} \Delta s_0 \\ \Delta q_0 \\ \Delta s_1 \\ \Delta q_1 \\ \Delta s_2 \\ \vdots \\ \Delta s_{N-1} \\ \Delta q_{N-1} \\ \Delta s_N \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$$

to eliminate $\Delta s_1, \ldots, \Delta s_N$ from QP.
Results in *condensed QP* in variables Δs_0 and $\Delta q_0, \ldots, \Delta q_N$ only.

Riccati Recursion

Alternative to condensing: can use Riccati recursion within QP solver addressing the full, uncondensed, but block sparse QP problem.

- Same algorithm as discrete time Riccati difference equation
- Linear effort in number N of shooting nodes, compared to $O(N^3)$ for condensed QP.
- Use Interiour Point Method to deal with inequalities, or Schur-Complement type reduction techniques.

Summary

- Direct Single and Multiple Shooting solve equivalent NLPs, i.e. they have the same discretization errors.
- Multiple shooting keeps the initial states of all shooting intervals as optimization variables, while single shooting eliminates all states by a forward simulation.
- The Generalized Gauss-Newton method is advantageous in case of least-squares cost functions with small residuals