Exercise 2: Least-Squares Estimation

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1 Exercise

1.1 Minimizer

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$$\begin{aligned} f(x) &= \|Ax - b\|_2^2 \\ &= (Ax-b)^T(Ax-b) \\ &= (x^TA^T - b^T)(Ax-b) \\ &= x^TA^TAx - b^TAx - x^TA^Tb + b^Tb \end{aligned}$$

First Order condition: We derive by x in numerator layout (Jacobian formulation) to get:

$$\begin{aligned} \nabla_x f = \frac{\partial f}{\partial x} &= x^T(A^TA + (A^TA)^T) - b^TA - (A^Tb)^T \\ &= 2x^TA^TA - 2b^TA \stackrel{!}{=} 0 \end{aligned}$$

Resolving by x:

$$\begin{aligned} 2x^TA^TA - 2b^TA &= 0 \\ x^TA^TA - b^TA &= 0 \\ x^TA^TA &= b^TA \\ x^T &= b^TA(A^TA)^{-1} \\ x^* &= (A^TA)^{-1}A^Tb \end{aligned}$$

✓

The second derivation is then defined by:

$$\nabla_x^2 f = \frac{\partial^2 f}{\partial x^2} = 2A^TA$$

Second Order condition: x^* is a local minimum, iff $\nabla_x^2 f$ is positive semidefinite at x^*. Since $A^T A$ is a Gramian matrix, f is positve semidefinite. Thus, the second order condition is met. ✓

1.2 Randomized Minimizer

For the mean, it holds that:

$$\begin{aligned} \mathrm{E}[X^*] &= (A^T A)^{-1} A^T \mathrm{E}[Y] \\ &= (A^T A)^{-1} A^T \mu_b \end{aligned}$$

Further, with the results from exercise 1, we get for the covariance: ✓

$$\begin{aligned} \mathrm{cov}[X^*] &= (A^T A)^{-1} A^T \mathrm{cov}[Y]((A^T A)^{-1} A^T)^T \\ &= (A^T A)^{-1} A^T \mathrm{cov}[Y] A (A^T A)^{-1} \\ &= (A^T A)^{-1} A^T \Sigma_b A (A^T A)^{-1} \end{aligned}$$

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