Exercise 5: Linear Least Squares (advanced)

(to be returned on Nov 25, 2014, 8:15 in HS 26, or before in building 102, 1st floor, 'Anbau')

Prof. Dr. Moritz Diehl and Robin Verschueren

Please remember to provide a solution on paper (written or typed) including all the necessary graphs from MATLAB. The MATLAB code (.m-files) should be sent to
robin.verschueren@gmail.com and giovanni@ampyxpower.com

Aim of this sheet is to perform fitting of a non-linear curve with linear least squares and to become acquainted with recursive least squares.

Solution

1. **Modeling the car:**

   (a) The linear ODE is as follows:

$$\dot{v}_X(t) = C_1 D(t) - C_2 - C_3 v_X(t).$$

Solution with undetermined coefficients:
We are looking for the general solution as the sum of the solution to the homogeneous equation and a particular solution.

$$v_{\mathrm{g}}(t) = v_{\mathrm{h}}(t) + v_{\mathrm{p}}(t).$$

First, rewrite the ODE as such:

$$\dot{v}_X(t) + C_3 v_X(t) = C_1 D - C_2.$$

Let's assume a solution to the homogeneous equation ($\dot{v}_X + C_3 v_X = 0$) is of the form $A \cdot \exp(\lambda t)$. Putting this into the homogeneous differential equation gives us:

$$\lambda \cdot e^{\lambda t} + C_3 \cdot e^{\lambda t} = 0.$$

Crossing out the exponentials, because they are bigger than zero everywhere, we get $\lambda = -C_3$, leading to

$$v_h = A \cdot \exp(-C_3 t).$$

For the particular solution, we take $v_p = B$, as the right hand side is also constant (D is constant). Filling in the original differential equation gives us:

$$0 + C_3 B = C_1 D - C_2,$$
$$\text{or} \quad B = \frac{C_1 D - C_2}{C_3}.$$

Now we have a general solution:

$$v_{\mathrm{g}} = A e^{-C_3 t} + \frac{C_1 D - C_2}{C_3}.$$

We can determine the last unknown coefficient A from the inital condition $\{v_X(0) = 0\mathrm{m/s}\}$:

$$0 = A \cdot 1 + \frac{C_1 D - C_2}{C_3},$$
$$\text{or} \quad A = -\frac{C_1 D - C_2}{C_3},$$

which leads to

$$v_X(t) = \frac{C_1 D - C_2}{C_3} \cdot (1 - e^{-C_3 t}).$$

Solution with Laplace transform:
In the Laplace domain, the ODE is

$$sV_X(s) - v_X(0) + C_3 V_X(s) = \frac{C_1 D - C_2}{s},$$
$$V_X(s) = \frac{C_1 D - C_2}{s(s + C_3)} \cdot \frac{C_3}{C_3} + \frac{v_X(0)}{s + C_3}.$$

Transforming back to the time domain, we get

$$v_X(t) = \frac{C_1 D - C_2}{C_3}(1 - \exp(-C_3 t)) + v_X(0) \cdot \exp(-C_3 t),$$

and after filling in the initial condition, we get the same answer as above.

(b) A well known fact from mechanics is

$$p_X(t) = \int_0^t v_X(\tau)\,\mathrm{d}\tau + p_X(0),$$

which gives in our case:

$$p_X(t) = \frac{C_1 D - C_2}{C_3} \cdot [\tau]_0^t + \frac{C_1 D - C_2}{C_3^2} \cdot [\exp(-C_3 \tau)]_0^t - \frac{v_X(0)}{C_3} \cdot [\exp(-C_3 \tau)]_0^t + p_X(0),$$
$$= \frac{C_1 D - C_2}{C_3} \cdot t + \left(\frac{C_1 D - C_2}{C_3^2} - \frac{v_X(0)}{C_3} \right) (\exp(-C_3 t) - 1) + p_X(0).$$