Exercises for Lecture Course on Modelling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2014

Exercise 3: Another review of optimization, systems, and statistics

(to be returned on Nov 11, 2014, 8:15 in HS 26, or before in building 102, 1st floor, 'Anbau')

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Aim of this sheet is to review some basic mathematical principles from optimization, systems, and statistics that are needed in the rest of the course. It is easier than the previous exercise sheets.

Exercise Tasks

1. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x) = x^\top Q\, x + c^\top x$ with fixed $c \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ (not necessarily symmetric). Compute, for any x, the gradient $\nabla f(x) \in \mathbb{R}^n$ and the Hessian $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ of this function. If Q is symmetric and positive definite, can you compute the unique minimizer x^* and the minimum function value $f(x^*)$?

   Hint: Write a matrix-vector product $b = Qx$ as $b_i = \sum_{j=1}^{n} Q_{ij}\, x_j$, for $i = 1, \ldots, m$. (2 points)

2. Give the state-space model ($\dot{x} = Ax + Bu$ and $y = Cx$) for a moving point mass m [kg] on a frictionless horizontal plane, e.g. an ice hockey puck gliding on ice. A 2D-horizontal force F [N] acting on the point mass is to be considered as the control input u to the system, and the 2D-position of the puck is the measured output y [m]. Choose a suitable state vector x with appropriate dimensions, and write down the matrices A, B and C.

   Hints: Use Newton's second law of motion $F = m \cdot a$. To find the states, think about what quantities you need to know (besides the input) to predict the future positions of the puck. (2 points)

3. Download the data `data.txt` from
   `https://www.imtek.de/professuren/systemtheorie/lehre/ws14/modelling/uebung-2`,
   and import it in MATLAB. It is a long sequence of N random numbers $x(k) \in \mathbb{R}$ for $k = 1, \ldots, N$.

   (a) To get a first idea of the distribution, plot the data $x(k)$ as a function of the index k, and compute the mean and the variance of the data series. (1 point)

   (b) Plot a histogram using 200 bins of appropriate size. (1 point)

   (c) * If you like, try to find out from which distribution these samples might be drawn from. Guess a suitable probability density function and plot it into the histogram. Creative answers will be rewarded! (2 bonus points)

   (d) Now group the data into data blocks of size M. Compute the mean value of each block for $M = 5, 50, 100, 500$, and store the resulting mean block values as a vector of length N/M. Now regard these mean block values as a new sequence of random numbers. For each of the four new sequences, compute again mean and variance, and a histogram. *Tip: You might want to check that your code gives identical results as the histogram of the original data for $M = 1$.* (2 points)

   (e) How does the variance depend on M? Plot it as a function of M for $M = 1, 5, 50, 100, 500$. What form does the distribution have for the larger values of M? How is this related to the central limit theorem in statistics? (2 points)

   (f) * Come up with a solution for the calculations in (d) without for-loops. (2 bonus points)

This sheet gives in total 10 points and 4 bonus points