REVIEW:

DIVISION LINES IN DYN. SYSTEM MODELS:

- CONT. VS. DISCRETE TIME
  - FOR LTI SYSTEMS

- STATE SPACE VS. INPUT OUTPUT MODELS

ALL FOUR COMBIS ARE POSSIBLE, AND CAN BE TRANSFORMED INTO EACH OTHER

	CONT (LTI)	DISC (LTI)
STATE	$\dot{x} = f(x,u)$, $y = g(x,u)$ $\dot{x} = Ax + Bu$, $y = Cx + Du$	$x_{k+1} = f(x_k, u_k)$, $y_k = g(x_k, u_k)$ $x_{k+1} = A x_k + B u_k$, $y_k = C x_k + D u_k$
I/O	$y^{(n)} = f(y, \dot{y}, \ddot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(m)})$ $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = b_m u^{(m)} + \dots + b_0 u$	$y_{k+n} = f(y_k, y_{k-1}, \dots, y_{k-n}, u_k, u_{k-1}, \dots, u_{k-m})$ $a_1 y_k + a_2 y_{k-1} + \dots + a_n y_{k-n+1} = b_1 u_k + \dots$
TRANSFER FCN (ONLY FOR LTI)	$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_0}$	Z-TRANSFORM $G(z) = \frac{b_1 z^n + \dots + b_n}{a_1 z^n + \dots + a_n}$

GET THIS BY plugging in $y(t) = e^{st}$ INTO I/O EQ

$y_k = z^k$

- LINEAR VS. NONLINEAR

5.4 LINEAR-IN-THE-PARAMETERS MODELS (LIP)

GENERAL LIP FOR DISCR. TIME I/O-MODELS WITH PARS Θ.

$$y_{k+1} = \sum_{i=0}^{m} \Phi_i(y_k, y_{k-1}, \dots, y_{k-n}, u_k, \dots, u_{k-n}) \cdot \Theta_i$$

$\Phi_1, \dots, \Phi_m$: BASIS FUNCTIONS. EXAMPLES.

EX 1:

$\phi_1 = y_k$ $\quad \phi_{n+1} = u_k$

$\phi_2 = y_{k-1}$ $\quad \vdots$

$\vdots$ $\quad \phi_{2n} = u_{k-n}$

$\phi_n = y_{k-n+1}$

} LTI-SYSTEMS

$\theta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$

(SPECIAL EXAMPLE: $y_{k+1} = a \cdot y_k + b \cdot u_k$

$= \begin{bmatrix} \phi_1(y_k, u_k) \\ \phi_2(y_k, u_k) \end{bmatrix}^T \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

$\phi_1(y_k, u_k) = y_k$

$\phi_2(y_k, u_k) = u_{k+1}$

$\theta = \begin{bmatrix} a \\ b \end{bmatrix}$

EX 2: VOLTERRA MODELS: USE ALSO QUADRATIC BASIS FUNCTIONS

$y_{k+1} = a_1 y_k + a_2 y_k^2 + b_1 u_k + b_2 u_k^2 + c \cdot y_k \cdot u_k$

$= \begin{bmatrix} y_k \\ y_k^2 \\ u_k \\ u_k^2 \\ y_k u_k \end{bmatrix}^T \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c \end{bmatrix}$

ESTIMATION PARS. FOR LIP MODELS:

GIVEN: TIME SERIES $u_1, \ldots, u_N$ AND $y_1, \ldots, y_N$

MODEL STRUCTURE WITH BASIS FUNCTIONS

$\phi_1, \ldots, \phi_m$ OF $y_k, \ldots, y_{k-n}, u_k, \ldots, u_{k-n}$

AIM: GET θ THAT FITS THE DATA USING THE MODEL

$$y_{k+1} = \sum_{i=1}^{m} \phi_i(\cdot) \cdot \theta_i + \varepsilon_k$$

ε_k: NOISE ↔ "PREDICTION ERROR"

FORMULATE LLS:

$$\hat{\theta}_{LS} = \arg \min_{\theta} \sum_{k=n+1}^{N-1} \left(y_{k+1} - \sum_{i=1}^{m} \phi_i(y_k, \ldots, y_{k-n}, u_k, \ldots, u_{k-n}) \cdot \theta_i \right)^2$$

LLS, CAN SOLVE EASILY USING PSEUDO-INVERSE!

PREDICTION ERROR MINIMIZATION (PEM)

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} \qquad \varphi_k = \begin{bmatrix} \phi_1(y_k, y_{k-1}, \dots, u_k, \dots) \\ \vdots \\ \phi_m(y_k, \dots \quad, u_k, \dots) \end{bmatrix} \in \mathbb{R}^m$$

$$\min_\theta \sum_{k=n+1}^{N-1} (y_{k+1} - \varphi_k^T \cdot \theta)^2 \iff \min_\theta \left\| \underbrace{\begin{bmatrix} \varphi_{n+1}^T \\ \vdots \\ \varphi_{N-1}^T \end{bmatrix}}_{=:\Phi} \cdot \theta - \underbrace{\begin{bmatrix} y_{n+2} \\ \vdots \\ y_N \end{bmatrix}}_{=:Y} \right\|^2$$

$$\hat{\theta}_{LS} = \Phi^+ Y \qquad \text{DONE!}$$

EXAMPLE $\quad y_{k+1} = a\, y_k + b\, u_k \qquad y_1, \dots, y_N \qquad u_1, \dots, u_N \qquad N = 3 \qquad \theta = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\min_\theta \sum_{k=1}^{N-1} (y_{k+1} - a \cdot y_k - b\, u_k)^2 = (y_2 - a\, y_1 - b\, u_1)^2 + (y_3 - a\, y_2 - b\, u_2)^2$$

$$\varphi_1 = \begin{bmatrix} \phi_1(y_1, u_1) \\ \phi_2(y_1, u_1) \end{bmatrix} = \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \qquad \Phi = \begin{bmatrix} \varphi_1^T \\ \varphi_2^T \end{bmatrix} = \begin{bmatrix} y_1 & u_1 \\ y_2 & u_2 \end{bmatrix}$$

$$\left\| \Phi \cdot \theta - Y \right\|_2 = \left\| \begin{bmatrix} y_1 & u_1 \\ y_2 & u_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \right\|_2 \qquad Y = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$$

IF THE TRUTH IS INDEED EQUAL TO OUR ASSUMPTION

$$\boxed{y_{k+1} = \varphi_k^T \theta + \varepsilon_k}$$ WITH i.i.d. NOISE, GAUSSIAN

THIS IS A ML-ESTIMATOR, ALL THEORY FROM BEFORE APPLIES.

OTHER POSSIBILITIES FOR θ ESTIMATION EXIST, BUT USUALLY LEAD TO NONLINEAR LEAST SQUARES.

LIKE BEFORE: HAVE NOISY MEASUREMENTS $y_1, \dots, y_N$
$u_1, \dots, u_N$
(IN LAST TWO EXERCISES)

(*) $$y_{k+1}^{TRUE} = \varphi_k^{TRUE\,T} \cdot \theta$$

$$y_k = y_k^{TRUE} + \varepsilon_k^y$$
$$u_k = u_k^{TRUE} + \varepsilon_k^u$$

ESTIMATE

$$\bar{\theta} := \begin{bmatrix} y_1^{TRUE} \\ y_2^{TRUE} \\ \vdots \\ y_n^{TRUE} \\ \theta \\ u_1^{TRUE} \\ \vdots \\ u_N^{TRUE} \end{bmatrix}$$

BY MINIMIZING MISFIT

$$\min_{\bar{\theta}} \sum_{k=1}^{N-1} \left(y_{k+1} - y_{k+1}^{TRUE}(\bar{\theta})\right)^2 + \left(u_k - u_k^{TRUE}\right)^2$$

BIG SIMULATION ROUTINE USING (*)

VERY NONLINEAR IN $\bar{\theta}$

NONLINEAR LS